MATHEMATICAL SOLUTIONS OF THE ONE-DIMENSIONAL NEUTRON TRANSPORT EQUATION

Larry Thomas Davis

United States
Naval Postgraduate School

THESIS

MATHEMATICAL SOLUTIONS OF THE

ONE-DIMENSIONAL NEUTRON TRANSPORT EQUATION

by

Larry Thomas Davis

Thesis Advisor: C. Comstock
Thesis Advisor: G. A. Garrettson

June 1971

Mathematical Solutions of the One-Dimensional Neutron Transport Equation

by

Larry Thomas Davis
Ensign, United States Navy
B.S., United States Naval Academy, 1970

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE
(with major in Mathematics)

from the

NAVAL POSTGRADUATE SCHOOL
June 1971

ABSTRACT

Considering the case of one speed, steady state, isotropic scattering in homogeneous media with plane symmetry, this thesis developes the solution of the one-dimensional neutron transport equation by three separate techniques. The method of K. M. Case which makes use of the theory of generalized functions in forming a semi-classical eigenfunction expansion with both a continuous spectrum and a finite discrete spectrum is developed. Converting the neutron transport equation to an integral equation and then to a singular integral equation, a solution is found in a method due to T. W. Mullikin which has very useful convergence properties. Applying the method due to N. Weiner and E. Hopf to the integral equation form of the neutron transport equation, a solution is developed which depends heavily on complex variable theory. The similarities, differences, advantages and disadvantages in the three methods are pointed out, and specific example solutions are presented.

TABLE OF CONTENTS

I. INTRODUCTION

A. HISTORICAL SKETCH

The field of neutron transport deals primarily with attempts to describe the migration of neutrons through bulk media and utilization of these results in the designing of nuclear reactors. Theoretically, in order to obtain a complete description of the neutron population it is necessary to specify the distribution of neutrons in space, time, and velocity. This distribution function satisifies the linearized Boltzmann transport equation. This equation has been known for almost a century and until recently has been studied primarily in connection with radiative transfer. Only within the last thirty years have general methods been developed to solve transport problems exactly, and only then for linear one-dimensional problems with constant speed particles in homogeneous, isotropic scattering media. With the advent of nuclear power, interest in solutions to the Boltzmann equation shifted from problems in radiative transfer to neutron transport. For this reason the linearized Boltzmann equation is now generally referred to as the neutron transport equation [2,8].

A great deal of mathematical theory had to be developed before any of the methods for solving the Boltzmann equation could be found. This involved significant advances in various branches of mathematics. The work of N. I. Muskhelishvili in the area of singular integral equations and solutions of

the Hilbert problem enters heavily in the developments of these methods. The theory of distributions and generalized functions developed by L. Schwartz made possible a method employing a modified classical eigenfunction expansion. Finally, a delicate use of analytic function theory made possible a method based on the manipulation of Fourier transforms.

The form of the neutron transport equation which describes the class of one-velocity transport problems mentioned above is known as the "one-speed approximation" and in the multi-dimensional form can be written

$$\frac{\partial \Psi(\underline{r},\underline{\Omega},t)}{\partial t} + v\underline{\Omega}\cdot\underline{\nabla}\Psi + v\,\sigma(\underline{r},v)\Psi = S(\underline{r},\underline{\Omega},t) + v\,\sigma(\underline{r},v)\,c(\underline{r},v)\iint_{\Omega'}\Psi(\underline{r},\Omega',t)f(\Omega'\cdot\Omega,\underline{r},v)\,d^2\Omega' \tag{1.1}$$

where

$\psi(\underline{r},\underline{\Omega},t)d^3r\ d^2\Omega$ is the number of neutrons in the volume d^3r about $\underline{r}$ which move in the solid angle $d^2\Omega$ about $\underline{\Omega}$ at time t;

v is the speed of the neutrons;

$v\underline{\Omega}$ is the velocity;

$\sigma(\underline{r},v)$ is the total macroscopic cross section;

$c(\underline{r},v)$ is the average number of secondary neutrons produced by fission and scattering per collision;

$\frac{1}{v}S(\underline{r},\underline{\Omega},t)$ is the number of uncollided neutrons at position $\underline{r}$, traveling in direction $\underline{\Omega}$, at time t which have come from some external or internal source other than scattering; and

$f(\underline{\Omega}'\cdot\underline{\Omega},\underline{r},v)d\underline{\Omega}d\underline{\Omega}'$ is the scattering kernel and represents the probability that a neutron with initial direction in $d\underline{\Omega}'$ about $\underline{\Omega}'$, when scattered at $\underline{r}$, emerges from the collision with a direction vector within the solid angle $d\underline{\Omega}$ about $\underline{\Omega}$, [2,8].

The directional unit vector $\underline{\Omega}$ is given by $\underline{\Omega}(\theta,\phi)$, (Ref. Figure 1). For the specific case of one-dimensional steady state problems in a homogeneous, isotropic scattering media with plane symmetry and constant values for c, and "isotropic scattering" f, the one-speed approximation takes the form [2]

$$\mu \frac{\partial\psi(x,\mu)}{\partial x} + \sigma\psi(x,\mu) = \frac{c\sigma}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu' + S(x,\mu), \qquad (1.2)$$

where

$\mu = \cos\theta$, $-\infty \leq x \leq \infty$, and $-1 \leq \mu \leq 1$. This is the form of the neutron transport equation in which we will be primarily interested.

In 1931 N. Weiner and E. Hopf developed a technique for solving (1.2) using Fourier transforms and principles of analytic continuation. Weiner and Hopf applied their method to the one-dimensional (stellar) radiative transport problem. For a couple of simple but important cases they obtained elegant and useful answers, but as we shall see it was clear that to tackle any further problems in radiative transfer using this method would present formidable problems. For this reason there was relatively little done along the lines of solving (1.2) for a number of years.

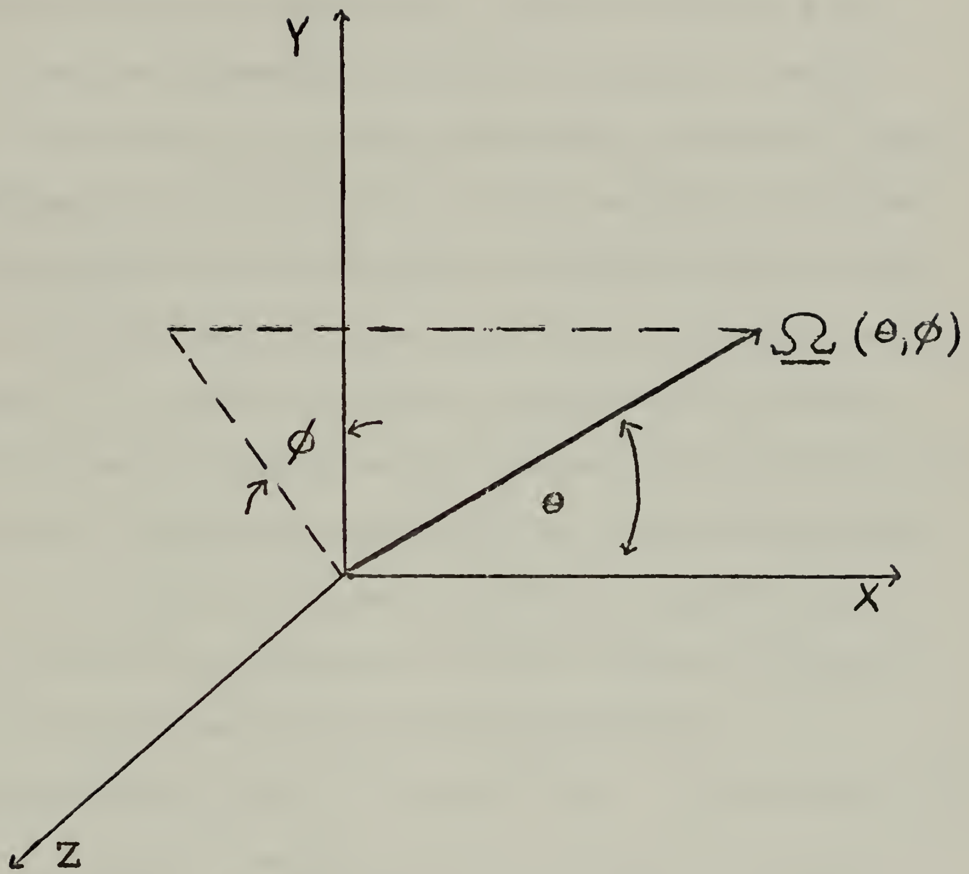

Figure 1. THE DIRECTIONAL UNIT VECTOR $\underline{\Omega}(\theta,\phi)$.

In 1953 the first comprehensive summary of neutron transport theory was written by K. M. Case, F. deHoffmann, and G. Placzek. A few years later B. Davison extended some of the efforts of Case, deHoffmann, and Placzek in a published work on one-dimensional neutron transport theory.

Then K. M. Case [2] in 1960 developed a powerful eigenfunction expansion technique for solving exactly one-dimensional, one-speed neutron transport problems based on the form (1.2) of the transport equation using properties of a Hilbert space. In order to devise a satisfactory theory Case had to go beyond the classical eigenfunction ideas and introduce both a continuous spectrum and a finite discrete spectrum. To do this he needed the tools of generalized functions developed by Schwartz and the theory of singular integral equations developed by Muskhelishvili.

In approximately 1963 A. Leonard and T. W. Mullikin [6] developed an alternate method by converting (1.2) to an integral equation in a novel fashion. This method appears to yield a more convenient solution for finite slab problems than does Case's method. It leans heavily on the theory of singular integral equations. In this thesis this integral approach is referred to as Mullikin's method.

There are various reasons for studying soluble one-speed problems, even for cases of initial physical assumptions as restricted as those imposed on equation (1.2). The physicist faced with solving a real physical problem is interested in two things. First, he wants to find out

what are the significant parameters of the problem, what factors influence the output, and how. For this type of analysis analytical expressions for the answer, even to approximate problems, form an invaluable tool. Second, he wants to be able to compute numerical results. For well understood problems a straightforward, if very tedious, numerical integration of the exact equations may be tried. For the problems in neutron transport, until recently the theory has been at the stage where it was much safer to try to obtain numerical answers from the analytical solutions. Unfortunately the analytical solutions are in the form of series and/or integrals whose convergence properties are minimal. Fortunately each of the methods mentioned above and described in detail below has a different region in which it gives satisfactory convergence for numerical, as opposed to purely mathematical, calculations. Finally, since the neutron transport equation is mathematically equivalent to the linearized Boltzmann equation, conclusions derived by studying it are directly applicable to transport problems in a variety of other fields such as astro-physics, plasma physics, radiation physics, and thermodynamics.

B. OBJECTIVES OF THE THESIS

In this thesis the methods of K. M. Case, T. W. Mullikin, and N. Weiner and E. Hopf are developed for solving the one-dimension, steady state, one-speed neutron transport equation in a homogeneous, isotropic scattering media with plane symmetry. Similarities among these three different approaches

to solving the transport problem are revealed. It is also shown how these three techniques eventually reduce to the same problem of solving a singular integral equation.

The novel feature of this thesis is in its simultaneous presentation of these three techniques and the comparison of them. Each of them attacks the problem from an entirely different mathematical approach. It is hoped this approach might enable the reader to choose the method that will yield the most convenient solution to his problem. We will try to make obvious the various advantages of one technique over another for solving a particular problem in transport theory. Finally, a knowledge of this material might provide insight into solving problems relating to more general physical models in neutron transport and other phenomena governed by the Boltzmann transport equation.

C. OUTLINE OF THE CHAPTERS

In Chapter II the method due to Case is developed. The existence of an eigenfunction expansion solution to the transport equation is assumed and values for discrete and continuum eigenvalues and eigenfunctions are determined. Using the usual property of orthogonality and assuming completeness of the eigenfunctions, the expansion coefficients are isolated and evaluated. As an aid to solving infinite half-space and finite slab problems a half-range orthogonality and a half-range completeness theorem are proved. The theory requires the use of the ideas of L. Schwartz and Plemelj, in addition to classical eigenfunction theory.

The method due principally to Mullikin is presented in Chapter III. Starting with the conventional integral equation form of (1.2) a reduction is made to a linear singular integral equation in terms of a complex parameter. The equation is then reduced to a Hilbert problem. The Hilbert problem is solved, yielding a solution to the initial integral equation. An approximation form of this solution using a truncated Neumann series is presented.

In Chapter IV the integral equation form of (1.2) is solved using the Weiner-Hopf technique. Assuming the existence of the Fourier transform of the functions contained in the basic integral equation and applying the principles of analytic continuation, the transform of the total neutron density function is determined.

Some observations and conclusions obtained through a study of the methods developed in Chapters II-IV are presented in Chapter V. Finally, example solutions by all three methods to specific problems are presented in the appendices.

II. THE METHOD DUE TO K. M. CASE

A. FORMULATION OF THE PROBLEM

Our problem is to find a solution to the equation

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1} \psi(x,\mu')\, d\mu' + S(x,\mu), \qquad (2.1)$$

where distance x is measured in units of mean free path so that $\sigma=1$. In the spirit of the method of solving ordinary differential equations we will first consider the properties of solutions to the homogeneous form of (2.1). This approach is an example of the application of the Fredholm Alternative Theorem where we investigate the form of solutions to the homogeneous equation to determine a general solution for the nonhomogeneous equation.

Let us look for solutions of the form

$$\psi_\nu(x,\mu) = \phi_\nu(\mu)\, e^{-x/\nu}, \quad -1 \le \mu \le 1, \; -\infty \le x \le \infty, \qquad (2.2)$$

and refer to ν and ϕ_ν as the eigenvalues and eigenfunctions of the solution $\psi_\nu(x,\mu)$. The impetus for solutions of this form comes from the usual eigenfunction expansion technique of solving ordinary differential equations. In that case exp(xp) is the form of the eigenfunction for ordinary differential operators with constant coefficients. We also observe that an eigenfunction expansion of the form (2.2) is equivalent to taking the Laplace transform

$$L\{\cdot\} = \int_0^\infty (\cdot)\, e^{-sx}\, dx, \qquad (2.3)$$

where we have $s=1/\nu$.

Substituting our proposed solution into (2.1) yields

$$(1 - \mu/\nu)\, \phi_\nu(\mu) = \frac{c}{2} \int_{-1}^{1} \phi_\nu(\mu')\, d\mu' \, . \tag{2.4}$$

Since this is a linear homogeneous equation for ϕ_ν, we are free to choose our normalization,

$$\int_{-1}^{1} \phi_\nu(\mu')\, d\mu' = 1 \quad . \tag{2.5}$$

Applying this to (2.4) yields the equation for ϕ_ν,

$$\phi_\nu(\mu) = \frac{c\nu}{2} \; \frac{1}{\nu - \mu} \quad . \tag{2.6}$$

In classical theory (2.6) is the only possible solution for $\phi_\nu(\mu)$. However, if one allows the possibility of having "distributions" ("generalized functions") in the sense of L. Schwartz, this is not the complete solution. The vanishing coefficient in (2.4) occurring when $\nu=\mu$ suggests the use of generalized functions in forming a complete solution for $\phi_\nu(\mu)$. With this in mind let us now write (2.6) as

$$\phi_\nu(\mu) = \frac{c\nu}{2} \, \frac{1}{\nu - \mu} + \lambda(\nu)\, \delta(\nu - \mu) \, , \tag{2.7}$$

where $\lambda(\nu)$ is an arbitrary function of ν. To check the validity of (2.7) we substitute it into (2.6) and find that

$$(\nu - \mu)\, \phi_\nu(\mu) = \frac{c\nu}{2}\left(\frac{\nu}{\nu-\mu} - \frac{\mu}{\nu-\mu}\right) + (\nu-\mu)\, \lambda(\nu)\, \delta(\nu-\mu) = \frac{c\nu}{2} \, . \tag{2.8}$$

Since the Dirac delta function is defined so that $x\delta(x) \equiv 0$, adding the term $\lambda(\nu)\delta(\nu-\mu)$ to (2.6) does not alter it as a solution to (2.4). Since we will be interested in integrals of $\phi_\nu(\mu)$ over ν and μ, rather than $\phi_\nu(\mu)$ itself, we may also,

without loss of generality, make the term $1/\nu-\mu$ in (2.7) the Cauchy principal value $P\,\frac{1}{\nu-\mu}$. The resulting solution to (2.4) then becomes

$$\phi_\nu(\mu) = \frac{c\nu}{2} P \frac{1}{\nu-\mu} + \lambda(\nu)\,\delta(\nu-\mu) . \qquad (2.9)$$

It is noted that since $-1\le\mu\le1$ when ν takes on values outside the real interval [-1,1], there is no possibility that ν will equal μ and (2.9) reduces to (2.6). Also, the eigenfunctions defined by (2.9) are not functions in the usual sense, but are instead "distributions" in the sense of Laurent Schwartz.

B. EVALUATION OF DISCRETE EIGENVALUES AND EIGENFUNCTIONS

Let us initially consider all ν such that $\nu\notin[-1,1]$. The function (2.6) satisfies the equation (2.4) for arbitrary ν. As usual, however, we expect only for certain values ν, possibly complex, will (2.6) satisfy the requirements of the problem, which include the normalization (2.5). To find these values of ν we integrate both sides of (2.6) with respect to μ and obtain the result

$$\Lambda(\nu) = 0 \quad , \qquad (2.10)$$

where

$$\Lambda(\nu) \equiv 1 - \frac{c\nu}{2}\int_{-1}^{1} \frac{d\mu}{\nu-\mu} \quad , \qquad \nu \text{ complex.} \qquad (2.11)$$

Therefore, to determine the eigenvalues ν for which (2.10) holds we need to find the zeros of $\Lambda(\nu)$.

We initially note that $\Lambda(\nu)$ may also be written in the following forms:

$$\Lambda(\nu) = 1 - \frac{c\nu}{2} \ln\left(\frac{\nu+1}{\nu-1}\right) = 1 - \frac{c\nu}{2} \ln\left(\frac{1+1/\nu}{1-1/\nu}\right), \; \nu \notin [-1,1], \quad (2.11a)$$

or

$$\Lambda(\nu) = 1 - c\nu \tanh^{-1} 1/\nu \;, \quad \nu \notin [-1,1] \,. \quad (2.11b)$$

We now state the following theorem about the properties of $\Lambda(\nu)$.

Theorem 2.1:

The function $\Lambda(\nu)$ defined by (2.11) has the properties

a. $\Lambda(\nu)$ is analytic in the complex ν plane cut from -1 to +1;

b. $\Lambda(-\nu) = \Lambda(\nu)$;

c. If ν is a zero of $\Lambda(\nu)$, so is the complex conjugate $\bar{\nu}$; and

d. $\Lambda(\nu)$ has only two zeros in the complex ν plane cut from -1 to +1, and they lie on the real or imaginary axes.

Proof:

The analyticity of Λ is apparent from the integral expression (2.11), [9]. The existence of a cut in the complex ν plane becomes obvious if we write (2.11a) in the form

$$\Lambda(\nu) = 1 - \frac{c\nu}{2} \ln\left\{\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}\right\} = 1 - \frac{c\nu}{2} \ln\frac{r_1}{r_2} - \frac{c\nu}{2} i(\theta_1-\theta_2), \quad (2.12)$$

where $\nu+1=r_1e^{i\theta_1}$ and $\nu-1=r_2e^{i\theta_2}$, and let ν approach the cut $[-1,1]$ from above and below.

The second property may be verified directly by using (2.11a). Therefore, if ν is a zero of (2.11), so is $-\nu$.

The third property may be verified by taking the complex conjugate of the expression $\Lambda(\nu)=0$.

Our final property is not as obvious as the others. From the Argument Theorem [3] we know that

$$\Delta_C \arg \Lambda(\nu) = 2\pi(N_z - N_p) \;, \tag{2.13}$$

where N_z and N_p are the number of zeros and poles, respectively, within any contour C. To find the number of zeros in the entire plane, take the contour C as in Figure 2. Since $\Lambda(\nu)$ is analytic in the cut plane, $N_p=0$. Applying equation (2.11b) and expanding $\tanh^{-1}\frac{1}{\nu}$ in a Taylor series yields

$$\Lambda(\nu) = 1 - c\nu\left[\tfrac{1}{\nu} + O(\tfrac{1}{\nu})^3\right] \;, \tag{2.14}$$

which implies $\Lambda(\nu)\to 1-c$ as $\nu\to\infty$. Therefore, $\Lambda(\nu)$ is a constant at infinity (i.e. no argument change at infinity), and we need only consider the contour around the cut. We shall need the following result due to Plemelj [12], which we will not prove.

Theorem 2.2:

Let a function $Q(\alpha)$ be defined for any complex α not on a cut C_1 by

$$Q(\alpha) = \frac{1}{2\pi i}\int_{C_1} \frac{\phi(s)}{s-\alpha}\, ds \;, \tag{2.15}$$

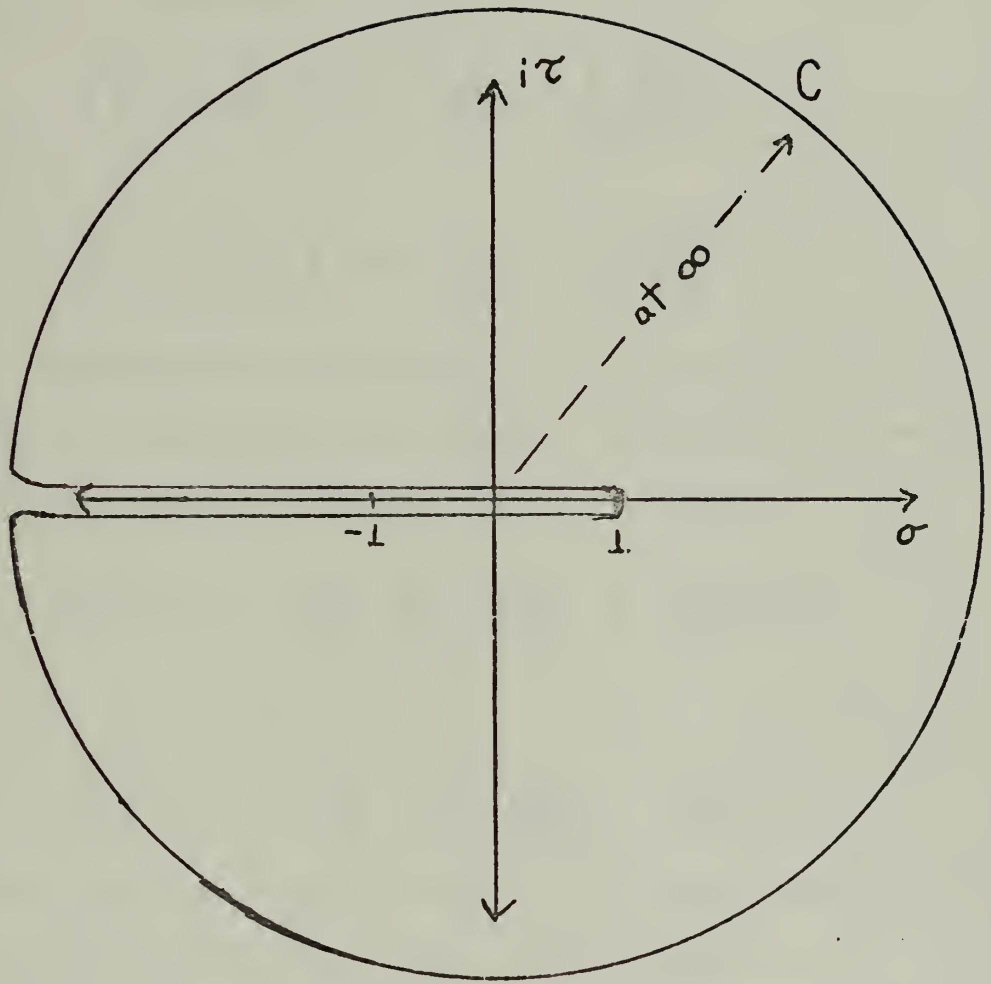

Figure 2.2. ARGUMENT THEOREM: $\Delta_C \arg\Delta(\nu) = 2\pi(N_z - N_p)$.

where $\phi(s)$ satisfies a Hölder condition on C_1. Denote the limiting value of $Q(\alpha)$ as α tends to a point t of C_1 from above by $Q^+(t)$, and from below by $Q^-(t)$. Then $Q^+(t)$ and $Q^-(t)$ are given by

$$Q^+(t) = \frac{1}{2}\,\phi(t) + \frac{1}{2\pi i}\; P\int_{C_1} \frac{\phi(s)}{s-t}\,ds \tag{2.15a}$$

and

$$Q^-(t) = -\frac{1}{2}\,\phi(t) + \frac{1}{2\pi i}\; P\int_{C_1} \frac{\phi(s)}{s-t}\,ds \quad . \tag{2.15b}$$

Consider now the form of $\Lambda(\nu)$ given by (2.11). If we allow μ to approach ν on the cut from above and below and apply the Plemelj formulas above we obtain

$$\Lambda^+(\nu) = 1 - \frac{c\nu}{2}\; P\int_{-1}^{1} \frac{d\mu}{\nu-\mu} + \frac{c\nu}{2}\,\pi i \tag{2.16a}$$

and

$$\Lambda^-(\nu) = 1 - \frac{c\nu}{2}\; P\int_{-1}^{1} \frac{d\mu}{\nu-\mu} - \frac{c\nu}{2}\,\pi i \quad . \tag{2.16b}$$

We see that as ν varies from 0 to 1 along the cut, $\Lambda^+(\nu)$ decreases monotonically from 1 to $-\infty$, implying that $\arg\Lambda^+(\nu)=0$, $\arg\Lambda^+(1)=\pi$, and hence $\Delta_{0,1}\arg\Lambda^+=\pi$. Repeating this process around the cut we find that $\Delta_C\arg\Lambda(\nu)=4\pi$, so by the Argument Theorem $N_z=2$.

We now attempt to draw some conclusions concerning the location of these two zeros. A way of solving $\Lambda(\nu)=0$ amenable to graphical methods is to find the points where the two functions $f=1/c$ and $g(\nu)=\tanh^{-1}\frac{1}{\nu}$ intersect. Properties b and c above tend to suggest that we might initially look

for those zeros on the axes of the complex plane. Therefore, we shall consider the following two cases.

Case 1: $c<1$; ν restricted to real values. From Figure 3 we observe that $\pm\nu_0$ are solutions to $\Lambda(\nu)=0$, and that $\nu_0>1$ and $-\nu_0<1$.

Case 2: $c>1$; ν restricted to purely imaginary values. Setting $\nu=i\tau$ we find that

$$\Lambda(\nu) = 1/c - \tau \tan^{-1} 1/\tau \quad . \tag{2.17}$$

From Figure 4 we observe that when $c>1$, $\Lambda(\nu)$ will have two purely imaginary zeros, $\pm\nu_0=\pm i\tau_0$.

For $c=1$ we see that from Case 1 that $\nu_0\to\infty$ as $c\to 1$. From Case 2 we conclude the same result since $1/\tau_0\to 0$ as $c\to 1$.

On the basis of the preceeding arguments we conclude that $\Lambda(\nu)$ has only two zeros in the complex plane cut at $[-1,1]$, and that having found two by inspection we have them all. They are real when $c<1$, and they are purely imaginary when $c>1$. When $c=1$ there exists a double zero at infinity. This completes our proof.

These zeros for $\Lambda(\nu)$, $\pm\nu_0\notin[-1,1]$, are called the DISCRETE EIGENVALUES. The corresponding DISCRETE EIGENFUNCTIONS are denoted

$$\phi_{0\pm}(\mu) = \pm \frac{c\nu_0}{2} \frac{1}{\pm\nu_0-\mu} \quad . \tag{2.18}$$

C. EVALUATION OF CONTINUUM EIGENVALUES AND EIGENFUNCTIONS

Let us now consider all ν such that $\nu\in[-1,1]$. The eigenfunctions here take the form of (2.9). Integrating both

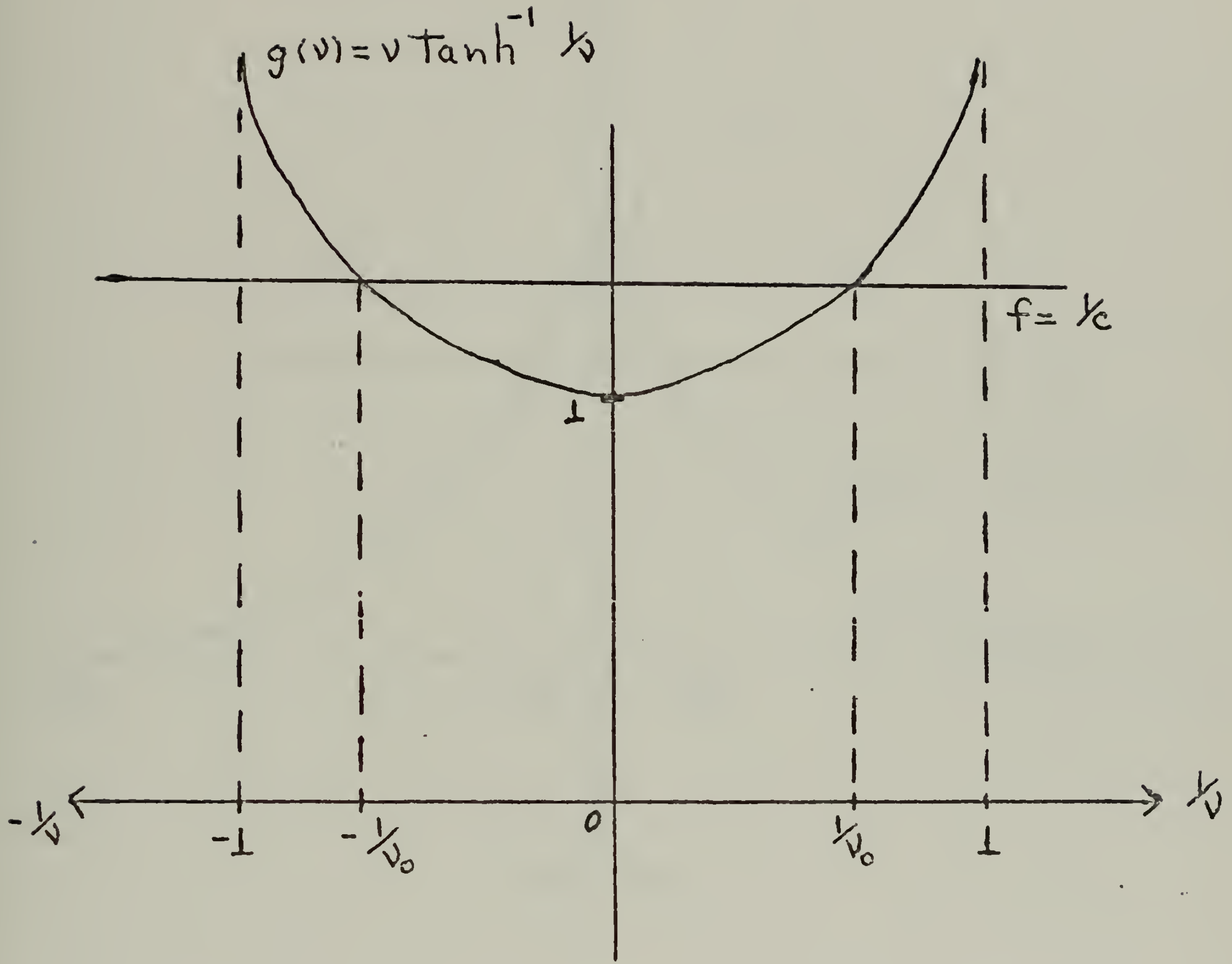

Figure 3. ZEROS OF $\Lambda(\nu)$ FOR $c<1$, ν REAL.

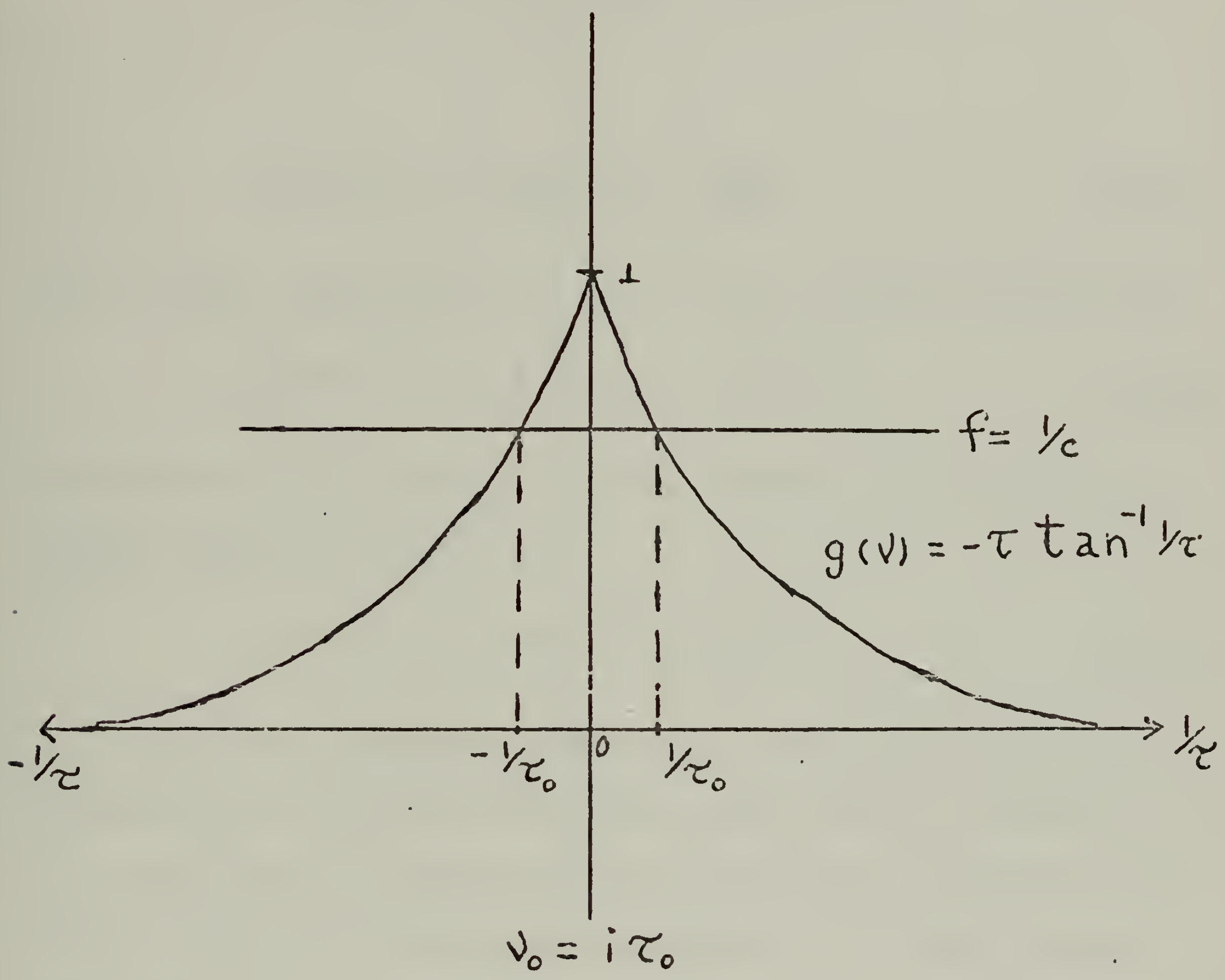

Figure 4. ZEROS OF $\Lambda(\nu)$ FOR $c>1$, ν PURELY IMAGINARY.

sides of (2.7) with respect to μ and applying our normalization, we have

$$1 = \frac{c\nu}{2} \, P\int_{-1}^{1} \frac{d\mu}{\nu-\mu} + \lambda(\nu)$$

or

$$\lambda(\nu) = 1 - \frac{c\nu}{2} \, P\int_{-1}^{1} \frac{d\mu}{\nu-\mu} \qquad (2.19)$$

From (2.16a) and (2.16b) we obtain the alternate expression

$$\lambda(\nu) = \tfrac{1}{2}\left[\Lambda^{+}(\nu) + \Lambda^{-}(\nu)\right] . \qquad (2.19a)$$

Evaluation of the principal value integral in (2.19) gives another expression for $\lambda(\nu)$,

$$\lambda(\nu) = 1 - c\nu \tanh^{-1}\nu . \qquad (2.19b)$$

Since $\lambda(\nu)$ is defined for every value of ν on $[-1,1]$, in constrast to (2.10) which holds only for two values of λ, a continuum of eigenfunctions is given by equation (2.9) and any one of the equivalent forms for $\lambda(\nu)$ given above. The values $\nu \varepsilon [-1,1]$ and the corresponding eigenfunctions (2.9) are called the CONTINUUM EIGENVALUES and CONTINUUM EIGENFUNCTIONS, respectively.

D. FULL-RANGE ORTHOGONALITY, NORMALIZATION, AND COMPLETENESS PROPERTIES

1. Orthogonality of the Discrete Eigenfunctions

Having obtained the discrete and continuum eigenvalues and eigenfunctions we would like to employ the eigenfunctions in the "usual" way, i.e. to expand solutions to the transport equation in terms of them. Before this is

possible, it is necessary to prove that the eigenfunctions are orthogonal with some weight function, determine their normalization constants, and prove that they form a complete set, in the sense that an arbitrary function of μ can be expanded in terms of them. We shall first consider the problem of orthogonality.

Theorem 2.3 (Orthogonality):

The functions $\phi_\nu(\mu)$ are orthogonal with weight function μ in the sense that

$$\int_{-1}^{1} \mu\, \phi_\nu(\mu)\, \phi_{\nu'}(\mu)\, d\mu = 0 \quad , \nu \neq \nu'. \tag{2.20}$$

Proof:

Applying (2.3) for ν and ν' we have

$$(1 - \mu/\nu)\, \phi_\nu(\mu) = \frac{c}{2} \int_{-1}^{1} \phi_\nu(\mu')\, d\mu' \tag{2.21}$$

and

$$(1 - \mu/\nu')\, \phi_{\nu'}(\mu) = \frac{c}{2} \int_{-1}^{1} \phi_{\nu'}(\mu')\, d\mu' \quad . \tag{2.22}$$

If we multiply (2.21) by $\phi_{\nu}'(\mu)$, multiply (2.22) by $\phi_\nu(\mu)$, subtract the resulting equations, and integrate over μ, we find that

$$\left[1/\nu' - 1/\nu\right] \int_{-1}^{1} \mu\, \phi_\nu(\mu)\, \phi_{\nu'}(\mu)\, d\mu = 0 .$$

This proves the orthogonality theorem.

2. Normalization Constants

The next step is to compute the normalization integral when $\nu'=\nu$ in (2.21). For the discrete eigenfunctions this becomes

$$N_{0\pm} = \int_{-1}^{1} \mu \, \phi_{0\pm}^2(\mu) \, d\mu = \left(\frac{c \nu_0}{2}\right)^2 \int_{-1}^{1} \frac{\mu \, d\mu}{(\pm \nu_0 - \mu)^2} \, . \tag{2.23}$$

Using equation (2.11) it is easy to obtain the alternate expression

$$N_{0+} = \frac{c}{2} \nu_0^2 \left. \frac{\partial \Lambda(\nu)}{\partial \nu} \right|_{\nu=\nu_0} = \frac{c}{2} \nu_0^3 \left[\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2} \right] . \tag{2.24}$$

Setting $\nu_0 = -\nu_0$ we see that

$$-N_{0-} = N_{0+} \tag{2.25}$$

Evaluating the normalization integral for the continuum eigenfunctions is not as straightforward as that for the discrete eigenfunctions. Looking ahead, we know that we want to use the normalization integrals to evaluate the coefficients in the expansion of an arbitrary function $f(\mu)$,

$$f(\mu) = \int_{-1}^{1} A(\nu') \, \phi_{\nu'}(\mu) \, d\nu' \, . \tag{2.26}$$

To evaluate the expansion coefficient $A(\nu)$, we multiply both sides by $\mu \phi_\nu(\mu)$ and integrate to obtain

$$\int_{-1}^{1} \mu \, \phi_\nu(\mu) f(\mu) \, d\mu = \int_{-1}^{1} \mu \, \phi_\nu(\mu) \, d\mu \int_{-1}^{1} A(\nu') \, \phi_{\nu'}(\mu) \, d\nu' . \tag{2.27}$$

Evaluating the right hand side of (2.27) will not be straightforward since the distributions (generalized functions) are not square-integrable and Fubini's theorem cannot be used to change the order of integration. Defining the right side of (2.27) as the product of $A(\nu)$ with the normalization constant $N(\nu)$, we have

$$N(\nu) \equiv \frac{1}{A(\nu)} \int_{-1}^{1} \mu\, \phi_\nu(\mu)\, d\mu \int_{-1}^{1} A(\nu')\, \phi_{\nu'}(\mu)\, d\nu' \; . \tag{2.28}$$

Since the singularity of the functions $\phi_\nu(\mu)$ prohibit changing the order of integration in (2.28) we must evaluate (2.28) in an alternate manner.

First let us define a function that we <u>can</u> evaluate

$$\bar{N}(\nu) \equiv \frac{1}{A(\nu)} \int_{-1}^{1} d\nu' A(\nu') \int_{-1}^{1} \mu\, \phi_{\nu'}(\mu)\, \phi_\nu(\mu)\, d\mu \; , \tag{2.29}$$

and attempt to relate it to the unknown function $N(\nu)$. Applying form (2.9) of $\phi_\nu(\mu)$ we have

$$\bar{N}(\nu) = \frac{1}{A(\nu)} \int_{-1}^{1} d\nu' A(\nu') \int_{-1}^{1} \mu \; d\mu \Big[\lambda(\nu')\lambda(\nu)\,\delta(\mu-\nu')\,\delta(\mu-\nu) + \lambda(\nu')\,\delta(\mu-\nu')\,\frac{c\nu}{2}\, P\frac{1}{\nu-\mu} + \lambda(\nu)\,\delta(\mu-\nu)\,\frac{c\nu'}{2}\, P\frac{1}{\nu'-\mu} + \left(\frac{c}{2}\right)^2 \nu'\nu\, P\frac{1}{\nu'-\mu}\, P\frac{1}{\nu-\mu}\Big] \tag{2.30}$$

Applying the identity

$$\mu\left[\frac{1}{\nu'-\mu}\right]\left[\frac{1}{\nu-\mu}\right] \equiv \left[\frac{\nu}{\nu-\mu} - \frac{\nu'}{\nu'-\mu}\right]\frac{1}{\nu'-\nu} \tag{2.31}$$

to the fourth integral in (2.30), we obtain

$$\begin{aligned} A(\nu)\bar{N}(\nu) = \nu A(\nu)\lambda^2(\nu) &- \frac{c\nu}{2}\, P\int_{-1}^{1} \frac{A(\nu')\lambda(\nu')}{\nu'-\nu}\, \nu'\, d\nu' \\ &+ \frac{c\nu}{2}\,\lambda(\nu)\, P\int_{-1}^{1} \frac{A(\nu')}{\nu'-\nu}\, d\nu' \\ &+ \left(\frac{c}{2}\right)^2 \nu\, P\int_{-1}^{1} \frac{\nu' A(\nu')}{\nu'-\nu}\, d\nu' \; P\int_{-1}^{1} d\mu \left[\frac{\nu}{\nu-\mu} - \frac{\nu'}{\nu'-\mu}\right] . \end{aligned} \tag{2.32}$$

Using (2.19) we see that the last integral in (2.32) is

$$P\int_{-1}^{1} d\mu \left[\frac{\nu}{\nu-\mu} - \frac{\nu'}{\nu'-\mu}\right] = \frac{2}{c}\left[\lambda(\nu') - \lambda(\nu)\right] \; . \tag{2.33}$$

Inserting (2.33) back into (2.32) reduces that equation to

$$\bar{N}(\nu) = \nu\, \lambda^2(\nu) \; . \tag{2.34}$$

We are now faced with the problem of relating $N(\nu)$ to $\bar{N}(\nu)$. Applying (2.9) in (2.28) as we did above in (2.29), we find that the resulting first three terms of $A(\nu)N(\nu)$ are identical to the first three terms of $A(\nu)\bar{N}(\nu)$ in (2.32). The last term differs only in the order of integration. To relate these last terms we shall need the following formula due to Poincaré-Bertrand, [9].

Theorem 2.4:

If an arbitrary function $g(\mu,\nu')$ exists and satisfies a Hölder condition on the cut [-1,1], then

$$P\int_{-1}^{1}\frac{1}{\mu-\nu}\,d\mu\;\;P\int_{-1}^{1}\frac{1}{\nu'-\mu}\,g(\mu,\nu')\,d\nu' =$$
$$-\pi^2 g(\nu,\nu)+\int_{-1}^{1}d\nu'\int_{-1}^{1}P\frac{1}{\mu-\nu}\;P\frac{1}{\nu'-\mu}\,g(\mu,\nu')\,d\mu. \qquad (2.35)$$

Applying this result we obtain

$$A(\nu)\,N(\nu) = A(\nu)\,\bar{N}(\nu) + \frac{c^2\pi^2}{4}\,\nu^3\,A(\nu)\,, \qquad (2.36)$$

from which we get

$$N(\nu) = \nu\left[\lambda^2(\nu) + \frac{c^2\pi^2}{4}\,\nu^2\right]. \qquad (2.37)$$

If we use equations (2.16) and (2.17) we obtain the alternate expression

$$N(\nu) = \nu\,\Lambda^{+}(\nu)\,\Lambda^{-}(\nu)\,. \qquad (2.38)$$

Returning to (2.26) we see that

$$A(\nu) = \frac{1}{N(\nu)}\int_{-1}^{1}\mu\,\phi_\nu(\mu)\,f(\mu)\,d\mu. \qquad (2.39)$$

3. Full-Range Completeness

Before proving any theorem on completeness it will be necessary to define the class of expandable functions for which the completeness theorem will hold. This is accomplished by referring to the results of N. I. Muskhelishvili on singular integral equations.

Assume we have an integral equation for an unknown function f(t) of the form

$$g(t) = A(t)\, f(t) + P \int_a^b \frac{B(t')\, f(t')}{t' - t}\, dt' , \qquad (2.40)$$

where g(t), A(t), and B(t) are known. The sufficient conditions for the existence of a solution are the following.

Theorem 2.5:

Let g(t), A(t), B(t) satisfy Hölder conditions, where for example,

$$|g(t) - g(t')| < (\text{constant})|t-t'|^{\gamma} \qquad (2.41)$$

for $a<t,\ t'<b$ and $\gamma>0$, and

$$|g(t) - g(c)| < (\text{constant})/|t-c|^{\alpha} , (\alpha<1), \qquad (2.42)$$

for c an endpoint (a or b) and $a<t<b$. Also, if $A(t) \pm \pi i B(t) \neq 0$ in $[a,b]$, then solutions to (2.40) exists, satisfy Hölder conditions, and may be found in a manner similar to that described in the following completeness theorem.

We now have the following theorem, which we will prove.

Theorem 2.6 (Full-Range Completeness):

The eigenfunctions $\phi_{0\pm}(\mu)$ and $\phi_\nu(\mu)$ are complete for arbitrary functions $\psi(\mu)$ of the above mentioned class on the full range $-1\leq\mu\leq 1$ in the sense that

$$\psi(\mu) = a_{0+}\phi_{0+}(\mu) + a_{0-}\phi_{0-}(\mu) + \int_{-1}^{1} A(\nu)\,\phi_\nu(\mu)\,d\mu\,, \quad (2.43)$$

where a_{0-}, a_{0+}, and $A(\nu)$ are the expansion coefficients.

To prove this theorem we shall show that these expansion coefficients can be uniquely determined.

In proving this theorem we assume initially that $A(\nu)$ is itself a member of the expandable class of functions described above, and hence it is possible to write a function $\psi'(\mu)$ in the form

$$\psi'(\mu) = \int_{-1}^{1} A(\nu)\,\phi_\nu(\mu)\,d\nu, \quad (2.44)$$

If we can show that a solution to the singular integral equation (2.44) exists, then the expansion (2.43) does also. In the course of this proof two necessary conditions will be imposed on (2.44) for a solution to exist, and these conditions will enable us to find $a_{0\pm}$ so that (2.43) will exist also.

If we use equations (2.9) and (2.19a) in (2.44), we find that the equation reduces to

$$\psi'(\mu) = \frac{1}{2}\left[\Lambda^{+}(\mu)+\Lambda^{-}(\mu)\right]A(\mu) + \frac{c}{2}P\int_{-1}^{1}\frac{\nu A(\nu)}{\nu-\mu}\,d\nu. \quad (2.45)$$

Let us now define the following function

$$n(z) \equiv \frac{1}{2\pi i} \int_{-1}^{1} \frac{c}{2} \frac{\nu A(\nu)}{\nu - z} d\nu \qquad , \text{ z complex. } (2.46)$$

Under our assumption that $A(\nu)$ was a member of the class of expandable functions we see that $n(z)$ has the following properties:

a. $n(z)$ is analytic in the complex plane cut from -1 to 1;

b. $n^{+}(z) + n^{-}(z) = \frac{1}{\pi i} P \int_{-1}^{1} \frac{c\nu}{2} \frac{A(\nu)}{\nu - z} d\nu;$

c. $n^{+}(z) - n^{-}(z) = \frac{cz}{2} A(z), \; z \varepsilon [-1,1].$

Using b and c in (2.45) we obtain

$$\frac{c\mu}{2} \psi(\mu) = \frac{i}{2} \left[\Lambda^{+}(\mu) + \Lambda^{-}(\mu)\right]\left[n^{+}(\mu) - n^{-}(\mu)\right] + \frac{\pi i c\mu}{2}\left[n^{+}(\mu) + n^{-}(\mu)\right] . \qquad (2.47)$$

Returning to equations (2.16a) and (2.16b) we can obtain the useful relationship

$$\Lambda^{+}(\nu) - \Lambda^{-}(\nu) = \pi i c \nu \qquad (2.48)$$

which will enable us to reduce (2.47) to

$$\frac{c\mu}{2} \psi'(\mu) = \Lambda^{+}(\mu)\, n^{+}(\mu) - \Lambda^{-}(\mu)\, n^{-}(\mu) \; . \qquad (2.49)$$

Let us now define a new function

$$J(z) \equiv \Lambda(z)\, n(z) \; . \qquad (2.50)$$

Since $\Lambda(z)$ and $n(z)$ are analytic in the plane cut by $[-1,1]$, $J(z)$ is also. Therefore, our initial problem of solving

(2.44) for $\Lambda(\nu)$ has reduced to solving the following non-homogeneous Hilbert problem:

Find a function J(z), analytic in the complex plane cut from -1 to +1, such that

$$\frac{c\mu}{2}\,\psi'(\mu) = J^{+}(\mu) - J^{-}(\mu)\ ;\ \mu \varepsilon [-1,1]\ . \tag{2.51}$$

From the results due principally to Muskhelishvili the solution to this Hilbert problem is

$$J(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{c\mu}{2}\,\frac{\psi'(\mu)}{\mu - z}\,d\mu\ . \tag{2.52}$$

From this we determine the following expression for n(z),

$$n(z) = \frac{1}{\Lambda(z)}\,\frac{1}{2\pi i}\int_{-1}^{1} \frac{c\mu}{2}\,\frac{\psi'(\mu)}{\mu - z}\,d\mu\ . \tag{2.53}$$

Remembering that a requirement for this solution to exist was that n(z) be analytic in the cut plane and the fact that $\Lambda(z)$ vanishes when $z=\pm\nu_0$, we see that a solution given by (2.53) for n(z) will exist only if

$$\int_{-1}^{1} \frac{c\mu}{2}\,\frac{\psi'(\mu)}{\mu \pm \nu_0}\,d\mu = 0\ . \tag{2.54}$$

Therefore, we must now impose a condition on the expansion coefficients $a_{0\pm}$ so that (2.54) will hold.

Rewriting (2.43) in the following form

$$\psi(\mu) = \psi'(\mu) + a_{0+}\phi_{0+}(\mu) + a_{0-}\phi_{0-}(\mu)\ , \tag{2.55}$$

we see that (2.54) may be written

$$\int_{-1}^{1} \frac{c\mu}{2}\,\frac{\psi(\mu)}{\mu \pm \nu_0}\,d\mu = \int_{-1}^{1}\frac{\mu}{\mu\pm\nu_0}\left[a_{0+}\phi_{0+}(\mu) + a_{0-}\phi_{0-}(\mu)\right]d\mu \tag{2.56}$$

At this point we recall that the discrete eigenfunctions are given by

$$\phi_{0\pm} = \frac{c}{2} \frac{\nu_0}{\nu_0 \mp \mu} , \tag{2.57}$$

that the normalization constant is given by

$$N_{0\pm} = \int_{-1}^{1} \mu \phi_{0\pm}^2 (\mu) \, d\mu , \tag{2.58}$$

and that ϕ_{0+} is orthogonal to ϕ_{0-}. Applying these principles we note that (2.56) will be satisfied if we define

$$a_{0\pm} = \int_{-1}^{1} \mu \phi_{0\pm} (\mu) \psi(\mu) \, d\mu \Big/ N_{0\pm} \tag{2.59}$$

Therefore, if $a_{0\pm}$ is given by (2.59), (2.54) will be satisfied, n(z) will be analytic in the cut plane as desired, and property c for n(z) yields $A(\nu)$,

$$A(\nu) = \frac{2}{c\nu} \left[n^+(\nu) - n^-(\nu) \right] . \tag{2.60}$$

The full-range completeness theorem is thus proved.

4. Summary and Application

At this point let us summarize the results of sections A-D. Our problem is to find a solution to the homogeneous transport equation

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu') \, d\mu' . \tag{2.61}$$

We look for a solution of the form $\psi_\nu(x,\mu) = \phi_\nu(\mu) e^{-x/\nu}$ and determine that the discrete and continuum eigenfunctions are given by equations (2.18) and (2.9) with the appropriate definition for $\lambda(\nu)$. If the Laplace transform, $\bar{\psi}(\mu)$, of

$\psi(x,\mu)$ is a member of the class of expandable functions, we can expand it in terms of the eigenfunctions by (2.43), where the expansion coefficients are given by equations (2.59) and (2.60). The general solution to (2.61) is given by superposition of the eigenfunctions,

$$\psi(x,\mu) = a_{0+}\phi_{0+}(\mu)e^{-x/\nu_0} + a_{0-}\phi_{0-}(\mu)e^{-x/\nu_0} + \int_{-1}^{1} A(\nu)\phi_\nu(\mu)e^{-x/\nu}d\nu, \tag{2.62}$$

where the expansion coefficients $a_{0\pm}$ and $A(\nu)$ are determined by boundary conditions.

Let us now return to the problem of solving the inhomogeneous transport problem, given by equation (2.1), in a homogeneous infinite medium. If we can transform this problem into an equivalent homogeneous problem with appropriate boundary conditions, then the solution is given by equation (2.62). In practice this boundary condition might take the form of a "jump condition" at a neutron source, $q_s(x,\mu)$, distributed over a surface:

$$\mu\,(\psi_+ - \psi_-) = q_s(x,\mu). \tag{2.63}$$

ψ_+ and ψ_- represent the angular density on opposite sides of the surface over which the source $q_s(x,\mu)$ is distributed. In some problems $q_s(x,\mu)$ may be constructed to be equivalent to the combined effect of other distributed sources. In Appendix A a solution to the nonhomogeneous transport equation is developed using this approach for a particular example.

Another problem to consider is that of solving for the neutron distribution in a homogeneous half-space or a slab. In this case it is necessary to expand a function of μ defined not on the entire range, $-1\leq\mu\leq 1$, but rather on the half-range, $0\leq\mu\leq 1$. Obviously, our previous conclusions on orthogonality and completeness are not valid in this case, so we must develop half-range orthogonality and completeness properties to handle situations of this type. This is done in the following section. As an example of the application of these theorems the half-space albedo problem is solved in Appendix B.

E. HALF-RANGE COMPLETENESS AND ORTHOGONALITY PROPERTIES

Let us first consider the problem of half-range completeness.

Theorem 2.7 (Half-range Completeness):

> The functions $\phi_{0+}(\mu)$ and $\phi_\nu(\mu)$, $0\leq\nu\leq 1$, are complete for functions $\psi(\mu)$ of the expandable class of functions defined on the half-range $0\leq\mu\leq 1$.

The initial phase of the proof is analogous to that of full-range completeness in that we assume that a member $\psi'(\mu)$ of the expandable class of functions defined on the half-range, $0\leq\mu\leq 1$, can be expanded by

$$\psi'(\mu) = \int_0^1 A(\nu)\,\phi_\nu(\mu)\,d\nu\ , \quad 0\leq\mu\leq 1. \tag{2.64}$$

We now attempt to solve this singular integral equation for $A(\nu)$.

As before, substitution of the explicit form of $\phi_\nu(\mu)$ yields

$$\psi'(\mu) = \frac{1}{2}\left[\Lambda^+(\mu) + \Lambda^-(\mu)\right]A(\mu) + \frac{c}{2} P\int_0^1 \frac{\nu A(\nu)}{\nu - \mu}\, d\nu, \quad 0 \le \mu \le 1. \tag{2.65}$$

We now introduce the function

$$n(z) \equiv \frac{1}{2\pi i}\int_0^1 \frac{c}{2}\,\frac{\nu A(\nu)}{\nu - z}\, d\nu \;. \tag{2.66}$$

Assuming that $A(\nu)$ is a member of the expandable class of functions, we see that $n(z)$ has the following properties:

a. $n(z)$ is analytic in the complex plane cut from 0 to 1;

b. $n^+(z) + n^-(z) = \frac{1}{\pi i} P\int_0^1 \frac{c\nu}{2}\,\frac{A(\nu)}{\nu - z}\, d\nu$;

c. $n^+(z) - n^-(z) = cz/2\; A(z)$;

d. $n(z) \sim 1/z$ as $|z| \to \infty$.

Again following a similar procedure as before we can obtain the relation

$$\Lambda^+(\mu)\, n^+(\mu) - \Lambda^-(\mu)\, n^-(\mu) = \left[\frac{\Lambda^+(\mu) - \Lambda^-(\mu)}{2\pi i}\right]\psi'(\mu), \quad 0 \le \mu \le 1. \tag{2.67}$$

At this point we note that this Hilbert problem differs from the one in the previous theorem in that here n and Λ are not analytic over the same region. Λ is analytic in the complex plane cut from -1 to +1, but n is analytic in the complex plane cut from 0 to 1.

Rearranging equation (2.67),

$$n^+(\mu) - \frac{\Lambda^-(\mu)}{\Lambda^+(\mu)}\, n^-(\mu) = \left[\frac{\Lambda^+(\mu) - \Lambda^-(\mu)}{2\pi i\; \Lambda^+(\mu)}\right]\psi'(\mu) \;, \quad 0 \le \mu \le 1, \tag{2.68}$$

it is apparent that before solving for n(z) we must first solve the associated homogeneous Hilbert problem:

Find a function X(z), analytic and nonzero in the complex plane cut from 0 to 1, such that

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} \quad , \; 0 \leq \mu \leq 1 \; . \tag{2.69}$$

However, for the sake of continuity we will postpone solving for X(z) and proceed to solve for n(z) as though X(z) were known.

If we substitute our expression (2.69) into (2.68) we obtain after multiplying through by $X^+(\mu)$

$$n^+(\mu)\, X^+(\mu) - n^-(\mu)\, X^-(\mu) = \frac{\psi'(\mu)}{2\pi i}\left[X^+(\mu) - X^-(\mu)\right] \; , \; 0 \leq \mu \leq 1 \; . \tag{2.70}$$

In a similar manner as before we see that the solution to this Hilbert problem becomes

$$n(z)\, X(z) = \frac{1}{2\pi i} \int_0^1 \left[X^+(\mu) - X^-(\mu)\right] \frac{\psi'(\mu)}{\mu - z}\, d\mu \quad , \tag{2.71}$$

and our solution for n(z) is

$$n(z) = \frac{1}{X(z)} \cdot \frac{1}{2\pi i} \int_0^1 \left[X^+(\mu) - X^-(\mu)\right] \frac{\psi'(\mu)}{\mu - z}\, d\mu \; . \tag{2.72}$$

From property c we obtain the expression for A(ν) as

$$A(\nu) = \frac{2}{c\nu}\left[n^+(\nu) - n^-(\nu)\right] \; . \tag{2.73}$$

We now pause to investigate the explicit form of X(z) before proceeding to ensure that n(z) has the desired properties. Expanding the right hand side of (2.69) we obtain

$$\frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} = \frac{\lambda(\mu) + \frac{1}{2} c \pi i \mu}{\lambda(\mu) - \frac{1}{2} c \pi i \mu} = e^{i\theta(\mu)} \quad , \tag{2.74}$$

where

$$\theta(\mu) = 2 \arg \Lambda^{+}(\mu) = 2 \tan^{-1} \frac{c\pi\mu/2}{\lambda(\mu)} \tag{2.75}$$

A function which satisfies the ratio condition in (2.69) is

$$X_0(z) = \exp\left[\frac{1}{2\pi} \int_0^1 \frac{\theta(\mu')}{\mu' - z} \, d\mu'\right] . \tag{2.76}$$

The function X(z) we are seeking must also be analytic and nonzero in the plane cut from 0 to 1. From the form of $X_0(z)$ we can conclude that the only points where these properties might not hold are at the endpoints of integration when z=0 or z=1. Since $\lim_{\mu\to 0} \frac{\theta(\mu)}{\mu} = c\pi$ we need only investigate the endpoint at z=1.

Rewriting (2.76)

$$X_0(z) = \exp\left[\frac{1}{2\pi} \int_0^1 \frac{\theta - 2\pi}{\mu' - z} \, d\mu' + \ln \frac{z-1}{z}\right] , \tag{2.77}$$

yields

$$X_0(z) = \frac{z-1}{z} \exp\left[\frac{1}{2\pi} \int_0^1 \frac{\theta - 2\pi}{\mu' - z} \, d\mu'\right] . \tag{2.78}$$

Since

$$\lim_{\mu \to 1} \frac{\theta(\mu) - 2\pi}{\mu - 1} = \left.\frac{d\theta}{d\mu}\right|_{\mu=1} = 0 , \tag{2.79}$$

$X_0(z)$ has a zero at z=1. Therefore,

$$X(z) = \frac{1}{z-1} \exp\left[\frac{1}{2\pi} \int_0^1 \frac{\theta(\mu')}{\mu' - z} \, d\mu'\right] \tag{2.80}$$

satisfies all the required conditions.

We now proceed to ensure that n(z) satisfies all the required properties. Using the explicit form (2.80) for

X(z) in (2.72), it is apparent that n(z) will satisfy all the properties except d. Since $X(z) \sim 1/z$ as $|z| \to \infty$, d will be satisfied only if

$$\int_0^1 \left[X^+(\mu) - X^-(\mu)\right] \frac{\psi'(\mu)}{\mu - z}\, d\mu \sim 1/z^2 \quad \text{as } |z| \to \infty. \tag{2.81}$$

Since $1/\mu - z$ can be expanded to the form

$$\frac{1}{\mu - z} = -\frac{1}{z}\left[1 + \mu/z + \dots\right], \tag{2.82}$$

we see that relation (2.81) will be satisfied if

$$\int_0^1 \left[X^+(\mu) - X^-(\mu)\right] \psi'(\mu)\, d\mu = 0. \tag{2.83}$$

Remembering that we are attempting to expand a function $\psi(\mu)$ by

$$\psi(\mu) = \psi'(\mu) + a_{0+}\, \phi_{0+}(\mu), \tag{2.84}$$

we see that (2.83) will be satisfied if we choose

$$a_{0+} = \frac{\int_0^1 \psi(\mu)\left[X^+(\mu) - X^-(\mu)\right] d\mu}{\int_0^1 \left[X^+(\mu) - X^-(\mu)\right] \phi_{0+}(\mu)\, d\mu} \tag{2.85}$$

The theorem is thus proved. Summarizing, we observe that an arbitrary function $\psi(\mu)$ of the class of expandable functions for $0 \le \mu \le 1$ can be expanded in the form

$$\psi(\mu) = a_{0+}\, \phi_{0+}(\mu) + \int_0^1 A(\nu)\, \phi_\nu(\mu)\, d\nu, \tag{2.86}$$

where $a_{0\pm}$ is given by (2.85) and $A(\nu)$ is given by (2.73).

The orthogonality relation (2.20) of the previous section is not valid when the region of integration is restricted to $0 \le \mu \le 1$. Hence, to develop a half-range orthogonality

relation we must find a new weight function $W(\mu)$ such that

$$\int_0^1 W(\mu)\,\phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,d\mu = 0\,,\quad \nu\neq\nu'\,, \tag{2.87}$$

where $0\leq\nu$, $\nu'\leq 1$ or $\nu,\nu'=+\nu_0$.

Following the general procedure used to prove the full-range orthogonality theorem in the previous section, we take the defining equations for $\phi_\nu(\mu)$ and $\phi_{0+}(\mu)$

$$\left[1-\mu/\nu\right]\phi_\nu(\mu) = c/2\quad,\ 0\leq\nu\leq 1\,, \tag{2.88}$$

and

$$\left[1-\mu/\nu_0\right]\phi_{0+}(\mu) = c/2\quad, \tag{2.89}$$

multiply (2.88) by $W(\mu)\phi_{\nu}'(\mu)/\mu$, multiply the corresponding equation for $\phi_{\nu}'(\mu)$ by $W(\mu)\phi_\nu(\mu)/\mu$, subtract the two, and integrate the result over μ to get the relation

$$\begin{aligned}\left[1/\nu'-1/\nu\right]\int_0^1 W(\mu)\,\phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,d\mu &= \\ \frac{c}{2}\int_0^1\frac{W(\mu)}{\mu}\left[\phi_{\nu'}(\mu)-\phi_\nu(\mu)\right]d\mu\,;&\quad 0\leq\nu\,,\ \nu'\leq 1\,.\end{aligned} \tag{2.90}$$

Similarly,

$$\left[1/\nu_0-1/\nu\right]\int_0^1 W(\mu)\,\phi_\nu(\mu)\,\phi_{0+}(\mu)\,d\mu = \frac{c}{2}\int_0^1\frac{W(\mu)}{\mu}\left[\phi_{0+}-\phi_\nu\right]d\mu\,, \tag{2.91}$$

The orthogonality relation (2.87) will then be true only if

$$\int_0^1\frac{W(\mu)}{\mu}\,\phi_\nu(\mu)\,d\mu = \int_0^1\frac{W(\mu)}{\mu}\,\phi_{\nu'}(\mu)\,d\mu\,,\quad 0\leq\nu,\nu'\leq 1\,, \tag{2.92}$$

and if

$$\int_0^1\frac{W(\mu)}{\mu}\,\phi_\nu(\mu)\,d\mu = \int_0^1\frac{W(\mu)}{\mu}\,\phi_{0+}(\mu)\,d\mu\,. \tag{2.93}$$

Since (2.92) and (2.93) are to hold for all eigenvalues ν and ν', we can conclude that

$$\int_0^1 \frac{W(\mu)}{\mu} \phi_\nu(\mu)\, d\mu = A', \quad 0 \leq \nu \leq 1 \quad \text{or} \quad \nu = \nu_0, \tag{2.94}$$

where A' is a constant independent of ν. If we now substitute our explicit eigenfunction forms given by (2.18) and (2.9), we find that $W(\mu)$ must satisfy the singular integral equation

$$\frac{c\nu}{2} P\int_0^1 \frac{W(\mu)}{\mu(\nu-\mu)}\, d\mu + \frac{\lambda(\nu)\, W(\nu)}{\nu} = A', \quad 0 \leq \nu \leq 1, \tag{2.95}$$

and the equation

$$\frac{c\,\nu_0}{2} \int_0^1 \frac{W(\mu)}{\mu(\nu_0-\mu)}\, d\mu = A'. \tag{2.96}$$

Multiplying (2.95) by ν and (2.96) by ν_0 and applying the identity

$$\nu P\int_0^1 \frac{W(\mu)}{\mu(\nu-\mu)}\, d\mu \equiv P\int_0^1 \frac{W(\mu)}{\nu-\mu}\, d\mu + \int_0^1 \frac{W(\mu)}{\mu}\, d\mu \tag{2.97}$$

to the resulting form of (2.95) and similarly for the resulting form of (2.96), we obtain the following relations

$$\frac{c\nu}{2} P\int_0^1 \frac{W(\mu)}{\nu-\mu}\, d\mu + \lambda(\nu)\, W(\nu) = c\pi i \nu A, \tag{2.98}$$

and

$$\frac{c\,\nu_0}{2} \int_0^1 \frac{W(\mu)}{\nu_0-\mu}\, d\mu = c\pi i \nu_0 A, \tag{2.99}$$

where

$$A \equiv \frac{1}{c\pi i}\left[A' - \frac{c}{2}\int_0^1 \frac{W(\mu)}{\mu}\, d\mu\right] \tag{2.100}$$

As we did in the full-range completeness theorem, let us now define the function

$$m(z) \equiv \frac{1}{2\pi i} \int_c^1 \frac{W(\mu)}{\mu - z} \, d\mu \tag{2.101}$$

and rewrite equation (2.98) as

$$\Lambda^-(\nu)\, m^+(\nu) - \Lambda^+(\nu)\, m^-(\nu) = A\,(\Lambda^+ - \Lambda^-)\,, \quad 0 \le \nu \le 1\,. \tag{2.102}$$

Equation (2.99) then reduces to the auxiliary condition

$$m(\nu_c) = -A\,. \tag{2.103}$$

We note again that the problem of finding the weight function $W(\mu)$ has reduced to that of solving the nonhomogeneous Hilbert problem (2.101) for $m(z)$ such that

$$m^+(\nu) - \frac{\Lambda^+(\nu)}{\Lambda^-(\nu)}\, m^-(\nu) = \frac{A(\Lambda^+(\nu) - \Lambda^-(\nu))}{\Lambda^-(\nu)} \tag{2.104}$$

where, by (2.101), $m(z)$ is analytic in the complex plane cut from 0 to 1. We observe that since $\Lambda(\nu)$ is analytic in the cut plane from -1 to 1 and not 0 to 1, we must first solve the associated homogeneous Hilbert problem:

Find a function $Y(z)$, analytic in the complex plane cut from 0 to 1, such that

$$\frac{Y^+(\nu)}{Y^-(\nu)} = \frac{\Lambda^-(\nu)}{\Lambda^+(\nu)}\,, \quad 0 \le \nu \le 1\,. \tag{2.105}$$

Let us assume such a function $Y(z)$ exists. Substituting (2.105) into (2.104) and multiplying by $Y^+(z)$ we obtain

$$Y^+(z)\, m^+(z) - Y^-(z)\, m^-(z) = A\left[Y^-(z) - Y^+(z)\right]. \tag{2.106}$$

Temporarily disregarding the auxiliary condition (2.103) to be satisfied, we see after comparing this Hilbert problem to the one in the half-range completeness theorem that the functions Y(z) and X(z) are related in some manner similar to

$$Y(z) \sim 1/X(z) \qquad (2.107)$$

In order to satisfy the auxiliary condition let us try a Y(z) of the form

$$Y(z) = \frac{1}{X(z)} \frac{1}{\nu_0 - z} . \qquad (2.108)$$

Thus Y(z) will be analytic in the cut plane with the exception of a simple pole at $z=\nu_0$.

From (2.106) we conclude that

$$Y(z)\, m(z) = -\frac{A}{2\pi i}\int_0^1 \frac{Y^+(\mu') - Y^-(\mu')}{\mu' - z}\, d\mu' , \qquad (2.109)$$

Consider

$$E(z) \equiv Y(z)\, m(z) + \frac{A}{2\pi i}\int_0^1 \frac{Y^+(\mu') - Y^-(\mu')}{\mu' - z}\, d\mu' , \qquad (2.109a)$$

where E(z) is some entire function (except for the simple pole at ν_0) and by Liouville's theorem vanishes at infinity. Therefore, E(z) has the form

$$E(z) = \frac{C}{z - \nu_0} , \qquad (2.110)$$

where C is some constant to be determined. Equation (2.109) now becomes

$$m(z) = \frac{C}{z-\nu_0}\frac{1}{Y(z)} - \frac{1}{Y(z)}\frac{A}{2\pi i}\int_0^1 \frac{Y^+(\mu') - Y^-(\mu')}{\mu' - z}\, d\mu' . \qquad (2.111)$$

If we now apply Cauchy's integral formula to evaluate the integral of Y in (2.111), we obtain

$$m(z) = \frac{C}{Y(z)(z-\nu_0)} - A\left[1 + \frac{1}{X(\nu_0)(\nu_0 - z)Y(z)} - \frac{1}{Y(z)}\right]. \qquad (2.112)$$

Considering now our auxiliary condition given by (2.103) we can determine C from (2.110) as

$$C = -\frac{A}{X(\nu_0)}. \qquad (2.113)$$

Solving now for m(z) and W(μ) we obtain

$$m(z) = A\left[\frac{1}{Y(z)} - 1\right] \qquad (2.114)$$

and

$$W(\mu) = A(\nu_0 - \mu)\left[X^+(\mu) - X^-(\mu)\right], \qquad (2.115)$$

where A is an arbitrarily chosen constant. This establishes the orthogonality relation (2.87). Appendix B gives an example of how half-range orthogonality relations can be conveniently used to compute the expansion coefficients.

Reference Note:

The results of this chapter are due primarily to the work of K. M. Case and P. F. Zweifel [2].

III. THE METHOD DUE TO T. W. MULLIKIN

A. FORMULATION OF THE PROBLEM

Consider a homogeneous slab of thickness τ, with a total macroscopic cross section σ, and emitting c secondary neutrons per collision. The slab is infinite in the transverse directions and is surrounded by a non-reflecting medium, such as a vacuum or pure absorber. We assume that the one speed, steady state, and isotropic scattering conditions hold.

If we now define the total neutron density in the one-dimensional case as

$$\rho(x) = \int_{-1}^{1} \psi(x,\mu)\, d\mu , \tag{3.1}$$

the one-dimension transport equation (with $\sigma \equiv 1$)

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')\, d\mu' + S(x,\mu) \tag{3.2}$$

may be transformed to a Fredholm integral equation of the form

$$\rho(x) = \frac{c}{2} \int_{0}^{\tau} E_1(|x-x'|)\rho(x')\, dx' + \bar{S}(x) . \tag{3.3}$$

Here the assumption that no diffuse neutrons are reflected back into the slab becomes the boundary condition

$$\psi(0,\mu) = \psi(\tau,-\mu) = 0 \quad , 0 \le \mu \le 1 . \tag{3.4}$$

Using this condition we have applied the integrating factor $e^{-x'/\mu}$ to the differential equation (3.2) and have integrated

the resulting terms with respect to x' and μ. The kernel of integration is defined in the usual way as

$$E_1(|x-x'|) = \int_0^1 e^{-|x-x'|/\mu} \frac{d\mu}{\mu} . \tag{3.5}$$

Physically, the first term on the right side of equation (3.3) represents the neutron density due to neutrons scattered at least once before reaching point x. $\overline{S}(x)$ represents the neutron density due to uncollided neutrons from some external and/or internal source S(x,μ). For simplicity we consider the case of no internal sources, but an external source due to a unidirectional beam of neutrons incident at the x=0 surface with initial direction $\mu_0=\cos\theta_0$ measured from the normal (ref. Figure 5). In this case S(x,μ) takes the form

$$S(x,\mu) = \delta(x)\,\delta(\mu-\mu_0) \tag{3.6}$$

and $\overline{S}(x)$ becomes

$$\overline{S}(x) = e^{-x/\mu_0} , \tag{3.7}$$

Thus, we have reduced the problem of solving the differential equation (3.2) to the equivalent problem of solving the integral equation

$$\rho(x) = \frac{c}{2}\int_0^{\tau} E_1(|x-x'|)\rho(x')\,dx' + e^{-x/\mu_0} . \tag{3.8}$$

We might initially consider solving this Fredholm integral equation by a standard Neumann series approach. However, we observe that if the norm of the integral operator is

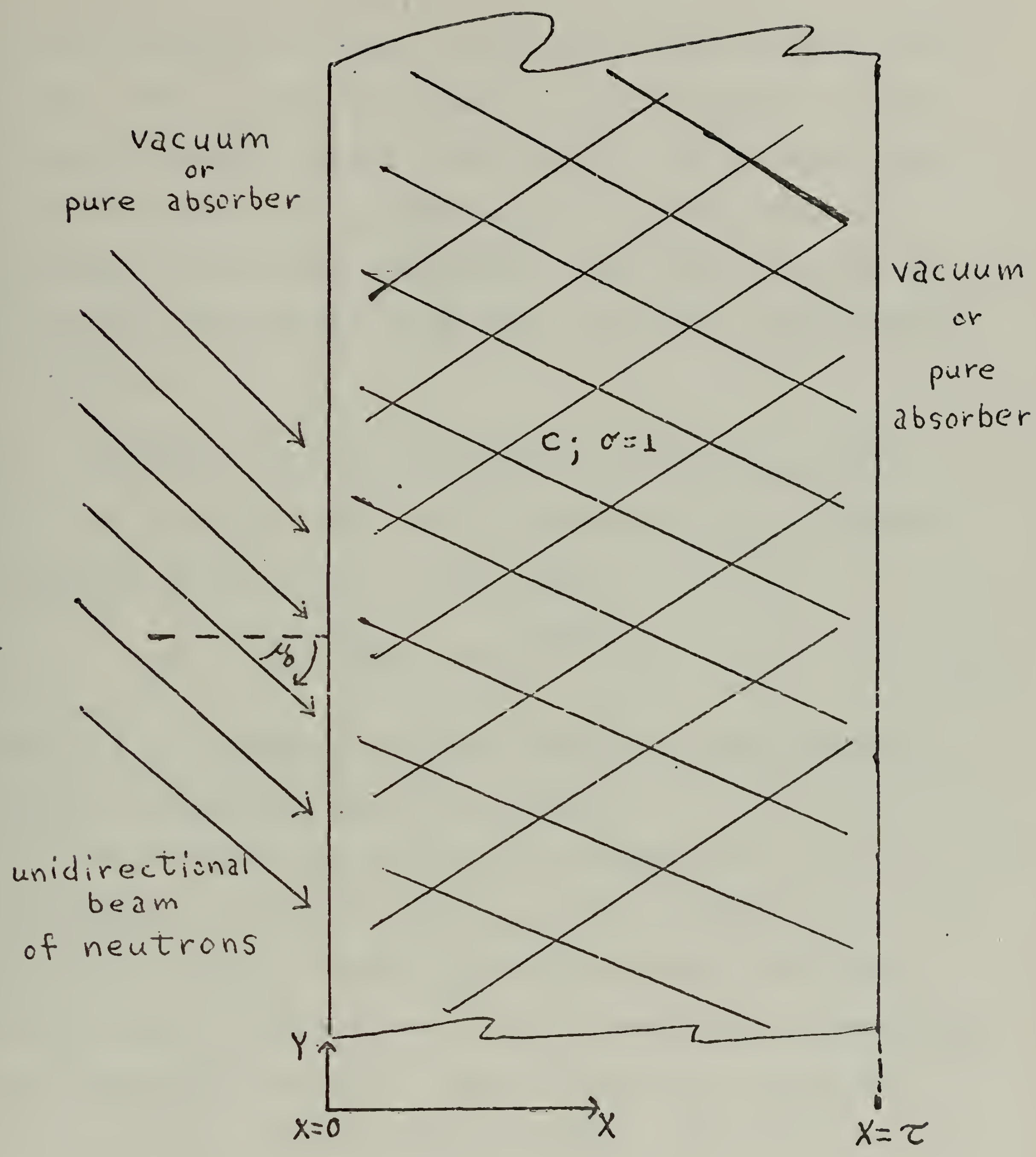

Figure 5. UNIDIRECTIONAL BEAM SOURCE OF NEUTRONS INCIDENT ON A BARE HOMOGENEOUS SLAB.

to the value 1, convergence will be quite slow. In practice this is usually the case for thick or near-critical ($c \sim 1$) slabs. Thus, while the Neumann series solution to (3.8) often converges rapidly enough in the case of thin slabs to be desirable, it is usually undesirable for thick slab problems due to slow convergence. We shall see that an alternate solution of (3.8) due to Mullikin removes this difficulty.

B. REDUCTION TO A LINEAR SINGULAR INTEGRAL EQUATION

At this point to simplify notation we will rewrite equation (3.8) in the operator form

$$\rho = \bar{c}\, T(\rho) + e^{-x/\mu_0}, \tag{3.9}$$

where T is a compact, positive definite, self-adjoint integral operator in $L_2[0,\tau]$ and $\bar{c} = c/2$.

We observe from (3.9) that ρ depends not only on x through the operator T but also on the parameter μ_0. Let us write (3.9) in a form (3.10) below which explicitly points out this dependence. For convenience in comparison with Mullikin's original paper we shall use the letter J for ρ and the complex number z for μ_0. We shall need the following result.

Theorem 3.1:

If T is the integral operator given above, and if $c \neq 1/\lambda_i$, where λ_i is an eigenvalue of T, then there exists a unique solution $J(x,z)$ to the equation

$$J(x,z) = \bar{c}\,T(J)(x,z) + e^{-x/z}, \tag{3.10}$$

which is analytic in the complex plane $|z|>0$.

As a sketch of the proof of Theorem 3.1 we observe that in (3.10) the operator (I-T) operates on J. By the Fredholm Alternative Theorem we see that if $\bar{c}$ is not equal to the reciprocal of an eigenvalue then a solution exists and can be written as the inverse operator $(I-T)^{-1}$ operating on the exponential function. The solution is linear and thus obviously unique. Since the inverse operator is an operator in x only, then the only dependence of the solution on z is the explicit dependence, and hence the solution is analytic in the complex plane $|z|>0$.

We shall make full use of the analytic dependence of J in (3.10) upon the parameter z and show that T(J) can be expressed in terms of integrals in the parameter z in J. We shall see that this result and the analyticity of J in the variable z lead to singular integral equations.

To obtain the equation for T(J) we apply the operator T to (3.10) and get

$$\begin{aligned} T(J)(x,z) - \bar{c}\,T(J)(x,z) &= T(e^{-x/z}) \\ &= e^{-x/z}\, z\int_{-1}^{1}\frac{d\mu}{z-\mu} - z\int_{0}^{1}\frac{e^{-x/\mu}}{z-\mu}\,d\mu - z\,e^{-\tau/z}\int_{0}^{1}\frac{e^{-(\tau-x)/\mu}}{z+\mu}\,d\mu . \end{aligned} \tag{3.11}$$

We shall also need the two relations

$$e^{-x/z} = (I-T)(e^{-x/z}) + T(e^{-x/z}) \tag{3.12}$$

and

$$J(x,z) = e^{-x/z} + (I-T)^{-1} T(e^{-x/z}) . \tag{3.13}$$

Combining (3.11), (3.12), and (3.13), we obtain the following result.

Theorem 3.2:

For $c \neq 1/\lambda_i$, the unique solution to (3.10) also satisfies

$$\begin{aligned} J(x,z) = e^{-x/z} &+ \frac{c}{2} z J(x,z) \int_{-1}^{1} \frac{dt}{z-t} - \frac{c}{2} z \int_{c}^{1} \frac{J(x,t)}{z-t} dt \\ &- \frac{c}{2} z e^{-\tau/z} \int_{c}^{1} \frac{J(\tau-x,t)}{z+t} dt . \end{aligned} \tag{3.14}$$

Equation (3.14) can also be written in the form

$$\begin{aligned} \Lambda(z) J(x,z) = e^{-x/z} &- \frac{c}{2} z \int_{c}^{1} \frac{J(x,t)}{z-t} dt \\ &- \frac{c}{2} z e^{-\tau/z} \int_{c}^{1} \frac{J(\tau-x,t)}{z+t} dt , \end{aligned} \tag{3.15}$$

where

$$\Lambda(z) = 1 - \frac{c}{2} z \int_{-1}^{1} \frac{dt}{z-t} = 1 - \frac{c}{2} z \ln\left(\frac{z+1}{z-1}\right) . \tag{3.16}$$

At this point we observe that we have reduced the problem of solving (3.3) for the total neutron density in terms of position x to that of finding a function J dependent upon x as a parameter and a complex variable z that satisfies the singular integral equation in z mentioned above.

C. SOLUTION OF THE SINGULAR INTEGRAL EQUATION

Instead of trying to solve equation (3.15) for $J(x,z)$, we shall find that it will be simplier to solve two equations

for two auxiliary functions $f_{\pm}$ connected to J by the following theorem.

Theorem 3.3:

If J(x,z) is a solution to equation (3.15), then the function $f_{(\pm)}(x,z)$ defined by

$$f_{(\pm)}(x,z) = J(x,z) \pm J(\tau - x, z) \tag{3.17}$$

is a solution to the equation

$$\begin{aligned}\Lambda(z)\, f_{(\pm)}(x,z) = \phi_{(\pm)}(x,z) + \frac{c}{2} z \int_0^1 \frac{f_{(\pm)}(x,t)}{t-z}\, dt \\ -(\pm)\, e^{-\tau/z} \frac{c}{2} \int_0^1 \frac{f_{(\pm)}(x,t)}{t+z}\, dt ,\end{aligned} \tag{3.18}$$

where

$$\phi_{(\pm)}(x,z) = e^{-x/z} \pm e^{-(\tau - x)/z} . \tag{3.19}$$

Assuming we can find $f_{(\pm)}(x,z)$, we can obtain J(x,z) by the simple relation

$$J(x,z) = \frac{1}{2}\left[f_{+}(x,z) + f_{-}(x,z)\right] . \tag{3.20}$$

We now make use of the analyticity of the functions in (3.18) in the variable z. Note that the first term on the right hand side of (3.18) is an entire function, the second is analytic everywhere in the complex plane except on (0,1), and the third is analytic everywhere except on (-1,0). If we let $\mathcal{J}_{(\pm)}(x,z)$ be the function consisting of the members of the right hand side of (3.18) that are analytic everywhere except on (-1,0), we obtain the expression

$$\mathcal{J}_{(\pm)}(x,z) = \phi_{(\pm)}(x,z) -(\pm)\, e^{-\tau/z} \frac{c}{2} z \int_0^1 \frac{f_{(\pm)}(x,t)}{t+z}\, dt . \tag{3.21}$$

Let us now make use of the following theorem from the theory of functions of complex variables [5,6,9].

Theorem 3.4:

If $f_{(\pm)}(x,t)$ satisfies a Hölder condition in the complex plane cut by (0,1), then the function

$$\Phi_{(\pm)}(x,z) = \frac{1}{2\pi i}\,\frac{c}{2}\int_0^1 \frac{f_{(\pm)}(x,t)}{t-z}\,dt \tag{3.22}$$

is analytic everywhere in the complex plane cut by (0,1). As in the derivation of Case's method, if we let $\Phi^{\pm}_{(\pm)}$ denote the limit of $\Phi_{(\pm)}$ as z approaches the cut (0,1) from above and below and apply the Plemelj conditions, we obtain [12]

$$\Phi^{\pm}_{(\pm)}(x,s) = \pm\frac{c\sigma}{4} f_{(\pm)}(x,s) + \frac{c\sigma}{2}\,\frac{1}{2\pi i}\,P\int_0^1 \frac{f_{(\pm)}(x,t)}{t-s}\,dt, \tag{3.23}$$

where $z \to s \varepsilon (0,1)$. From (3.18), (3.21), and (3.22) we obtain the expression

$$\Lambda(z) f_{(\pm)}(x,z) - 2\pi i z\,\Phi_{(\pm)}(x,z) = \mathcal{J}_{(\pm)}(x,z), \tag{3.24}$$

which due to the analyticity of $\mathcal{J}_{(\pm)}$ and $f_{(\pm)}$ on (0,1) becomes

$$\Lambda^{\pm}(s) f_{(\pm)}(x,s) - 2\pi i s\,\Phi^{\pm}_{(\pm)}(x,s) = \mathcal{J}_{(\pm)}(x,s),\; s\varepsilon(0,1). \tag{3.25}$$

Applying the Plemelj conditions to Λ we find

$$\Lambda^{\pm}(s) = 1 - \frac{c}{2}s\ln\left(\frac{1+s}{1-s}\right) \pm \frac{c\pi i}{2}s \;,\; s\varepsilon(0,1). \tag{3.26}$$

From (3.23) we observe that

$$\Phi^{+}_{(\pm)}(x,s) - \Phi^{-}_{(\pm)}(x,s) = \frac{c}{2} f_{(\pm)}(x,s) \;,\; s\varepsilon(0,1). \tag{3.27}$$

Using (3.26) and (3.27) in (3.25) yields the following result.

Theorem 3.5:

If $f_{(\pm)}(x,z)$ is a solution to (3.18), then $\Phi_{(\pm)}(x,z)$ defined by (3.22) is a solution to

$$\Lambda^{-}(s)\,\Phi^{+}_{(\pm)}(x,s) - \Lambda^{+}(s)\,\Phi^{-}_{(\pm)}(s) = \frac{c}{2}\,\mathcal{J}_{(\pm)}(x,s),\quad s\varepsilon(0,1),\tag{3.28}$$

where $\Lambda^{\pm}(s)$ is given by equation (3.26).

Thus, we have reduced our problem (3.18) to that of attempting to solve the nonhomogeneous Hilbert problem:

For x fixed, find a function $\Phi_{(\pm)}(x,z)$ analytic in the complex z plane cut by (0,1) and satisfying

$$\Phi^{+}_{(\pm)}(x,s) - \frac{\Lambda^{+}(s)}{\Lambda^{-}(s)}\,\Phi^{-}_{(\pm)}(x,s) = \frac{c}{2}\,\mathcal{J}_{(\pm)}(x,s)\Big/\Lambda^{-}(s),\quad s\varepsilon(0,1),\tag{3.29}$$

We observe that since $\Lambda(z)$ is analytic in the complex plane cut from -1 to 1, and $\Phi_{(\pm)}(x,z)$ is analytic in the complex plane cut from 0 to 1, it will first be necessary to solve the associated homogeneous Hilbert problem:

Find a function $X(z)$, analytic in the complex z plane cut by (0,1) with no zeros or poles, and satisfying the boundary condition

$$\frac{X^{+}(s)}{X^{-}(s)} = \frac{\Lambda^{+}(s)}{\Lambda^{-}(s)},\quad s\,\varepsilon\,(0,1).\tag{3.30}$$

We first determine the form of the right side of (3.30) by

$$\frac{\Lambda^{+}(s)}{\Lambda^{-}(s)} = \frac{\left(1-\frac{c}{2}s\ln\frac{1+s}{1-s}\right)+\frac{c\pi i s}{2}}{\left(1-\frac{c}{2}s\ln\frac{1+s}{1-s}\right)-\frac{c\pi i s}{2}} = e^{i\theta(s)}\tag{3.31}$$

where

$$\theta(s) = 2 \arg \Lambda^{+}(s) = 2 \tan^{-1}\left(\frac{c\pi s/2}{1-\frac{c}{2}\ln\frac{1+s}{1-s}}\right), \quad s\varepsilon(0,1). \quad (3.32)$$

Let us now look for a possible solution of the form

$$X(z) = K(z)\, e^{\phi(z)} \qquad (3.33)$$

where K(z) is a quotient of algebraic functions in z. The right side of equation (3.30) will now take the form

$$\frac{X^{+}(s)}{X^{-}(s)} = e^{\phi^{+}(s)-\phi^{-}(s)} = e^{i\theta(s)}, \quad s\varepsilon(0,1), \qquad (3.34)$$

and we are now faced with finding $\phi(z)$ such that

$$\phi^{+}(s) - \phi^{-}(s) = i\theta(s), \quad s\varepsilon(0,1). \qquad (3.35)$$

We now apply a theorem due to the results of Muskhelishvili [9].

Theorem 3.6:

If a complex function $\phi(z)$ satisfies the condition (3.35) on the cut (0,1), then the function is given by the equation

$$\phi(z) = \frac{1}{2\pi}\int_0^1 \frac{\theta(t)}{t-z}\,dt + P_1(z), \qquad (3.36)$$

where $P_1(z)$ is an arbitrary entire function of z. $\phi(z)$ is analytic in the complex plane cut by (0,1).

From Theorem 3.6 we now observe that X(z)/K(z) as defined by (3.33) and (3.36) will have no poles or zeros except possibly on the cut (0,1) or at the endpoints of the cut when approached from within the cut. Letting z approach the left

endpoint results in Θ approaching zero and we are left only with the right endpoint to investigate.

From (3.32) we notice that as z approaches 1 we will develop problems with our proposed form of X(z). Let us then write $\phi(z)$ as

$$\phi(z) = \frac{1}{2\pi}\int_0^1 \frac{\theta - 2\pi}{t - z}\,dt + \ln\frac{z-1}{z} + P_1(z) \quad . \tag{3.37}$$

We now observe that the first term on the right hand side of (3.37) is well behaved at both endpoints of the cut (0,1). Equation (3.33) will now become

$$X(z) = K(z)\,\frac{z-1}{z}\,\exp\left[\frac{1}{2\pi}\int_0^1 \frac{\theta - 2\pi}{t - z}\,dt\right] . \tag{3.38}$$

Therefore, to avoid the possibility of a zero in X(z) at z=1 we let K(z) take the form

$$K(z) = \frac{1}{z-1} \quad , \tag{3.39}$$

and we have established the following theorem.

Theorem 3.7:

If a function X(z) satisfies the conditions of the homogeneous Hilbert problem (3.30), then the function is given by

$$X(z) = \frac{1}{z-1}\exp\left[\frac{1}{2\pi}\int_0^1 \frac{\theta(t)}{t - z}\,dt\right] , \tag{3.40}$$

where we have taken $P_1=0$.

Having found a function X(z) satisfying (3.30) we substitute (3.30) into (3.29), divide through by $X^+(s)$ and obtain

$$\frac{\Phi^{+}_{(\pm)}(x,s)}{X^{+}(s)} - \frac{\Phi^{-}_{(\pm)}(x,s)}{X^{-}(s)} = \frac{c}{2}\,\frac{\mathcal{J}_{(\pm)}(x,s)}{\Lambda^{-}(s)\,X^{+}(s)}\,, \quad s\in(0,1). \tag{3.41}$$

From the results of Theorem 3.6 we know that the solution to this equation takes the form

$$\frac{\Phi_{(\pm)}(x,z)}{X(z)} = \frac{1}{2\pi i}\,\frac{c}{2}\int_0^1 \frac{\mathcal{J}_{(\pm)}(x,t)}{\Lambda^{-}(t)X^{+}(t)}\,dt + P_{2(\pm)}(x,z), \tag{3.42}$$

where $P_{2(\pm)}(x,z)$ is a polynomial in z with coefficients as functions of x.

By investigating the behavior of the functions in the left side of (3.42) as $|z|\to\infty$ we determine that $P_{2(\pm)}(x,z)$ will be a zero$\underline{\text{th}}$ order polynomial in z and hence only a function of x. If we write $P_{2(\pm)}(x,z)$ as

$$P_{2(\pm)}(x,z) \equiv -(\pm)\,\frac{C_{(\pm)}(x)}{2\pi i}\,, \tag{3.43}$$

we have the following result.

Theorem 3.8:

If a function $\Phi_{(\pm)}(x,z)$ satisfies the conditions of the nonhomogeneous Hilbert problem (3.29), then the function is given by

$$\Phi_{(\pm)}(x,z) = \frac{1}{2\pi i}\,X(z)\,\frac{c}{2}\int_0^1 \frac{\mathcal{J}_{(\pm)}(x,t)}{\Lambda^{-}(t)X^{+}(t)}\,\frac{dt}{t-z} - (\pm)\frac{1}{2\pi i}\,C_{(\pm)}(x)X(z). \tag{3.44}$$

At this point we can substitute (3.44) into (3.27) to obtain an expression for $f_{(\pm)}(x,z)$ in terms of $\mathcal{J}_{(\pm)}(x,z)$ or use another approach.

From (3.22) and (3.21) we have

$$2\pi i z\,\Phi_{(\pm)}(x,-z) = -(\pm)\,e^{\tau/z}\left[\mathcal{J}_{(\pm)}(x,z) - \mathcal{S}_{(\pm)}(x,z)\right]. \tag{3.45}$$

Substituting this equation into (3.44) we get

$$\mathcal{J}_{(\pm)}(x,z) = -(\pm)\, e^{-\tau/z}\, z\, X(-z)\, \frac{c}{2} \int_0^1 \frac{\mathcal{J}_{(\pm)}(x,t)}{X^{+}(t)\,\Lambda^{-}(t)}\, \frac{dt}{t+z}$$
$$+\, e^{-\tau/z}\, z\, X(-z)\, C_{(\pm)}(x) + \$_{(\pm)}(x,z)\,, \tag{3.46}$$

where $-z \notin (0,1)$. Equation (3.46) is a Fredholm integral equation for $J_{(\pm)}(x,z)$ whose Neumann series solution can be shown to converge rapidly.

To express $f_{(\pm)}(x,z)$ in terms of $J_{(\pm)}(x,z)$ we substitute (3.45) into (3.24) and obtain

$$\Lambda(z)\, f_{(\pm)}(x,z) = \mathcal{J}_{(\pm)}(x,z) + (\pm)\, e^{-\tau/z}\, \mathcal{J}_{(\pm)}(x,-z) - \$_{(\pm)}(x,z)\,. \tag{3.47}$$

Therefore, from (3.46) and (3.47) we can obtain a complete solution to equation (3.18). However, by proceeding a little further we can simplify the task of finding this solution.

Let us define the Fredholm integral operator L as

$$L\,[\cdot] \equiv \int_0^1 \frac{c}{2}\, \frac{e^{-\tau/z}\, z\, X(-z)}{X^{+}(t)\,\Lambda^{-}(t)}\, \frac{[\cdot]}{t+z}\, dt\,. \tag{3.48}$$

Equation (3.46) now becomes

$$\mathcal{J}_{(\pm)}(x,z) = -(\pm)\, L\,[\mathcal{J}_{(\pm)}](x,z) + \$_{(\pm)}(x,z) + e^{-\tau/z}\, z\, X(-z)\, C_{(\pm)}(x)\,. \tag{3.49}$$

Using the linear properties of L we can write $J_{(\pm)}(x,z)$ as

$$\mathcal{J}_{(\pm)}(x,z) = h_{1(\pm)}(x,z) + C_{(\pm)}(x)\, h_{2(\pm)}(z)\,, \tag{3.50}$$

where h_1 and h_2 are defined by the Fredholm equations

$$h_{1(\pm)}(x,z) = -(\pm)\, L\,[h_{1(\pm)}](x,z) + \$_{(\pm)}(x,z) \tag{3.51}$$

and

$$h_{2(\pm)}(z) = -(\pm)\, L\left[h_{2(\pm)}\right](z) + e^{-\tau/z} z\, X(-z)\,. \qquad (3.52)$$

Substituting (3.50) into (3.47) now yields an expression for $f_{(\pm)}$ in terms of $h_{1(\pm)}$ and $h_{2(\pm)}$

$$f_{(\pm)}(x,z) = \frac{1}{\Lambda(z)} \Big\{ h_{1(\pm)}(x,z) + (\pm)\, e^{-\tau/z}\, h_{1(\pm)}(x,-z) - \$_{(\pm)}(x,z) \qquad (3.53)$$
$$+ C_{(\pm)}(x) \left[h_{2(\pm)}(z) + (\pm)\, e^{-\tau/z}\, h_{2(\pm)}(-z) \right] \Big\}.$$

Let us now designate by z_0 a zero of the function $\Lambda(z)$. Comparing equation (3.16) with the definition of Λ in the development of Case's method we discover that the two functions are identical and hence z_0 corresponds to a discrete eigenvalue in that method. If we multiply (3.53) through by $\Lambda(z)$ and then use the fact that z_0 is a zero for $\Lambda(z)$ we get an expression for $C_{(\pm)}(x)$

$$C_{(\pm)}(x) = -\, \frac{h_{1(\pm)}(x,z_0) + (\pm)\, e^{-\tau/z_0}\, h_{1(\pm)}(x,-z_0) - \$_{(\pm)}(x,z_0)}{h_{2(\pm)}(z_0) + (\pm)\, e^{-\tau/z_0}\, h_{2(\pm)}(-z_0)} \qquad (3.54)$$

From definition (3.48) one can show [5,7] the Neumann series solutions to equations (3.51) and (3.52) will converge like $e^{-\tau}$ for large values of τ, and hence convergence is quite rapid for thick slab problems. Applying these solutions for $h_{1(\pm)}$ and $h_{2(\pm)}$ and equation (3.54), we see that (3.53) gives a complete solution for equation (3.18).

Summarizing, we see that we reduced the problem of finding the total neutron density $\rho(x)$ in equation (3.8) to the problem of solving a singular integral equation (3.15)

for a function $J(x,z)$, where z is an arbitrary complex valued parameter. Solving this singular integral equation was dependent upon solving a nonhomogeneous Hilbert problem. The solution to this Hilbert problem turned out to be very similar to that found in the proof of the half-range completeness theorem in the development of Case's method. The resulting solution to the singular integral equation was found in terms of Fredholm integral equations whose Neumann series solutions converge rapidly for large values of τ.

Having found a solution $J(x,z)$ for the singular integral equation, our desired solution $\rho(x)$ to the integral equation (3.8) immediately follows by setting $z=\mu_0$.

IV. SOLUTION BY THE WEINER-HOPF TECHNIQUE

A. FORMULATION OF THE PROBLEM

To solve neutron transport problems by the Weiner-Hopf technique we initially look at a modified form of the integral equation (3.8),

$$\rho(x) = S(x) + g(x) + \int_{-\infty}^{\infty} E_1(|x-x'|)\,\rho(x')\,dx' , \qquad (4.1)$$

where $\rho(x)$ and $S(x)$ are the familiar total neutron density and uncollided source functions. The function $-g(x)$ will represent the total neutron density due to sources and diffused neutrons on the negative side of some point $x=0$ if we restrict $\rho(x)$, $S(x)$, and $g(x)$ as

$$\left\{ \begin{array}{ll} \rho(x) = S(x) = 0 & , x<0 \\ g(x) = 0 & , x>0 . \end{array} \right\} \qquad (4.2)$$

Equation (4.1) now reduces to the pair of equations

$$\rho(x) = S(x) + \int_{0}^{\infty} E_1(|x-x'|)\rho(x')\,dx' \quad , x>0 \qquad (4.3)$$

$$0 = g(x) + \int_{0}^{\infty} E_1(|x-x'|)\rho(x')\,dx' , \quad x<0 . \qquad (4.4)$$

Therefore, equation (4.3) is exactly the original equation (3.8), and (4.4) extends (4.3) to the domain $(-\infty,\infty)$. In the slab problem equation (4.3) gives the total neutron density to the right of the $x=0$ face of the slab. The value of $-g(x)$ will be the total density to the left side of the $x=0$ face of the slab.

Assuming, *a priori*, that $\rho(x)$, $S(x)$, and $k(x)$ are in L_1, we now list the following Fourier transforms

$$P(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(x)\, e^{i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \rho(x)\, e^{i\xi x} dx \tag{4.5}$$

$$\mathcal{S}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S(x)\, e^{i\xi x}\, dx \tag{4.6}$$

$$K(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x)\, e^{i\xi x}\, dx \tag{4.7}$$

as the Fourier transforms of the total neutron density, of the uncollided source, and of the kernel function k(x) which is defined by

$$k(x-x') = \frac{\sqrt{2\pi}}{2}\, c\, E_1(|x-x'|) \;. \tag{4.8}$$

Since S and k are known functions, the assumption on ρ is the only one which we must verify. This we shall do later, when we have solved for ρ. If we let the function t(x) be defined by

$$t(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(x-x')\, \rho(x')\, dx', \tag{4.9}$$

the convolution theorem gives the familiar result,

$$T(\xi) = K(\xi)\, P(\xi) \tag{4.10}$$

where $T(\xi)$ is the Fourier transform of t(x),

$$T(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t(x)\, e^{i\xi x}\, dx \;. \tag{4.11}$$

With the same assumptions as above, we now take the Fourier transform of (4.1) and obtain

$$P(\xi) = \mathcal{S}_+(\xi) + G_-(\xi) + K(\xi)\, P(\xi), \tag{4.12}$$

where

$$\mathcal{S}_+(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty S(x)\, e^{i\xi x}\, dx \tag{4.13}$$

and

$$G_-(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 g(x)\, e^{i\xi x}\, dx \,. \tag{4.14}$$

By investigating the transform functions $P(\xi)$, $\mathcal{S}(\xi)$, and $G(\xi)$ we obtain the following result.

Theorem 4.1:

$P(\xi)$ and $\mathcal{S}_+(\xi)$ are analytic in the upper half complex plane, $\mathrm{Im}(\xi)>\alpha$, and $G_-(\xi)$ is analytic in the lower half-plane, $\mathrm{Im}(\xi)<\beta$. Equation (4.12) will in general hold in the strip $\alpha<\mathrm{Im}(\xi)<\beta$.

From (4.12) we have

$$\left[1 - K(\xi)\right] P(\xi) = \mathcal{S}_+(\xi) + G_-(\xi) \,, \quad \alpha < \mathrm{Im}(\xi) < \beta, \tag{4.15}$$

where $1\text{-}K(\xi)$ is found to be given by

$$1 - K(\xi) = 1 - \frac{c}{\xi} \arctan \xi \,, \tag{4.16}$$

using standard tables to compute $K(\xi)$ from (4.7). Unfortunately, (4.15) is one equation for two unknown functions P and G. In the Weiner-Hopf technique if we can find functions $K_+(\xi)$ and $K_-(\xi)$, analytic in the upper half-plane, $\mathrm{Im}(\xi)>\alpha$, and lower half-plane, $\mathrm{Im}(\xi)<\beta$, respectively, such that

$$1 - K(\xi) = \frac{K_+(\xi)}{K_-(\xi)} \,, \tag{4.17}$$

we can write (4.15) as

$$K_+(\xi)\,P(\xi) = K_-(\xi)\,\$_+(\xi) + K_-(\xi)\,G_-(\xi) \tag{4.18}$$

If in addition we can find two functions $R_+(\xi)$ and $Q_-(\xi)$, analytic in the upper half-plane, $\mathrm{Im}(\xi)>\alpha$, and lower half-plane, $\mathrm{Im}(\xi)<\beta$, respectively, such that

$$K_-(\xi)\,\$_+(\xi) = R_+(\xi) + Q_-(\xi) \quad , \tag{4.19}$$

we would be able to write (4.18) as

$$K_+(\xi)\,P(\xi) - R_+(\xi) = K_-(\xi)\,G_-(\xi) + Q_-(\xi)\,. \tag{4.20}$$

Since the functions

$$E_+(\xi) = K_+(\xi)\,P(\xi) - R_+(\xi) \tag{4.21}$$

and

$$E_-(\xi) = K_-(\xi)\,G_-(\xi) + Q_-(\xi) \tag{4.22}$$

provide the analytic continuation of each other to the whole complex plane and are equal on the strip $\alpha<\mathrm{Im}(\xi)<\beta$, then there is an entire function $E(\xi)$ equal to both (4.21) and (4.22). If we can determine the form of $E(\xi)$ we would be able to solve (4.21) for $P(\xi)$ and (4.22) for $G_-(\xi)$, and the inversions of these two transforms would result in expressions for our desired functions $\rho(x)$ and $g(x)$. We will need the following result from complex function theory [3].

Theorem 4.2:

A function that is analytic in a domain D is uniquely determined over D by its values along an arc interior to D.

Being an entire function, $E(\xi)$ is then uniquely determined by its behavior on some large circle. We shall determine this behavior.

B. DETERMINATION OF K_+, K_-, R_+, Q_-

In order to apply the ideas of A we must first determine the functions $K_+(\xi)$, $K_-(\xi)$, $R_+(\xi)$, and $Q_-(\xi)$. Let us first attack the problem of finding $K_+(\xi)$ and $K_-(\xi)$, analytic in the upper half-plane, $\mathrm{Im}(\xi)>\alpha$, and lower half-plane, $\mathrm{Im}(\xi)<\beta$, respectively, such that

$$\frac{K_+(\xi)}{K_-(\xi)} = 1 - K(\xi) = 1 - \frac{c}{\xi}\tan^{-1}\xi \, . \tag{4.23}$$

Since the branch lines for $c/\xi \tan^{-1}\xi$ are $[i,i\infty)$ and $[-i,-i\infty)$, we see that $1-K(\xi)$ is not analytic in either half-plane. In order to solve our problem we will have to take the natural logarithm of (4.23). We will then obtain

$$\ln K_+(\xi) - \ln K_-(\xi) = \ln(1-K(\xi)) = \ln\left(1-\frac{c}{\xi}\tan^{-1}\xi\right), \tag{4.24}$$

This introduces another pair of singularities, namely the zeros of $1-K(\xi)$, since at these points the function $\ln(1-K(\xi))$ fails to be analytic. Therefore, we look for a function $K^*(\xi)$ which does not have these zeros, is basically the same as $K(\xi)$ at 0 or ∞ and is otherwise analytic. Consider

$$\frac{K_+^*(\xi)}{K_-^*(\xi)} = \left[\frac{\xi^2+1}{\xi^2+z^2}\right](1-K(\xi)) \tag{4.25}$$

where $\pm iz$ are the (simple) zeros of $1-K(\xi)$. The numerator of the right hand side of (4.25) has zeros at the branch

points of $1-K(\xi)$ and thus does not add anything. If we take the natural logarithm of both sides of (4.25) we obtain

$$\ln K_+^*(\xi) - \ln K_-^*(\xi) = \ln\left[\left(\frac{\xi^2+1}{\xi^2+Z^2}\right)\left(1-\frac{c}{\xi}\tan^{-1}\xi\right)\right], \quad (4.26)$$

which has been constructed so that the logarithm of both sides is analytic in the strip $-1<\mathrm{Im}(\xi)<1$. Therefore, if we take a real number γ such that $0<\gamma<1$ and apply the Cauchy integral formula to (4.26), we find that

$$\ln K_+^*(\xi) - \ln K_-^*(\xi) = \frac{1}{2\pi i}\int_{-\infty-i\gamma}^{\infty-i\gamma} \ln\left[\left(\frac{s^2+1}{s^2+Z^2}\right)\left(1-\frac{c}{s}\tan^{-1}s\right)\right]\frac{ds}{s-\xi}$$
$$-\frac{1}{2\pi i}\int_{-\infty+i\gamma}^{\infty+i\gamma} \ln\left[\left(\frac{s^2+1}{s^2+Z^2}\right)\left(1-\frac{c}{s}\tan^{-1}s\right)\right]\frac{ds}{s-\xi} \quad (4.27)$$

The first integral on the right hand side of (4.27) is analytic for $\mathrm{Im}(\xi)>-\gamma$, and the second is analytic for $\mathrm{Im}(\xi)<\gamma$.

If we now set α in the above discussion equal to $-\gamma$ and β equal to γ, separate the functions in (4.27) and take their exponential values, and finally relate these results to $K_+(\xi)$ and $K_-(\xi)$ by (4.23) and (4.25), we obtain the desired forms

$$K_+(\xi) = (\xi^2+Z^2)\exp\left\{\frac{1}{2\pi i}\int_{-\infty-i\gamma}^{\infty-i\gamma} \ln\left[\left(\frac{s^2+1}{s^2+Z^2}\right)\left(1-\frac{c}{s}\tan^{-1}s\right)\right]\frac{ds}{s-\xi}\right\} \quad (4.28)$$

and

$$K_-(\xi) = (\xi^2+1)\exp\left\{\frac{1}{2\pi i}\int_{-\infty+i\gamma}^{\infty+i\gamma} \ln\left[\left(\frac{s^2+1}{s^2+Z^2}\right)\left(1-\frac{c}{s}\tan^{-1}s\right)\right]\frac{ds}{s-\xi}\right\}, \quad 0<\gamma<1, \quad (4.29)$$

the first of which is analytic in the upper half-plane, $\mathrm{Im}(\xi)>-\gamma$, and the second of which is analytic in the lower half-plane, $\mathrm{Im}(\xi)<\gamma$.

We are now ready to determine the functions $R_+(\xi)$ and $Q_-(\xi)$ such that equation (4.19) holds. Since $K_-(\xi)\, \mathscr{S}_+(\xi)$ is analytic in the strip $[-im, i]$, where $-im$ is the singularity in $\mathscr{S}_+$ (note $m>1$) we can apply the Cauchy integral formula again and get

$$K_-(\xi)\, \mathscr{S}_+(\xi) = \frac{1}{2\pi i} \int_{-\infty - i\gamma}^{\infty - i\gamma} \frac{K_-(s)\, \mathscr{S}_+(s)}{s-\xi}\, ds - \frac{1}{2\pi i} \int_{-\infty + i\gamma}^{\infty + i\gamma} \frac{K_-(s)\, \mathscr{S}_+(s)}{s-\xi}\, ds \,. \tag{4.30}$$

From this equation we see that $R_+(\xi)$ is given by

$$R_+(\xi) = \frac{1}{2\pi i} \int_{-\infty - i\gamma}^{\infty - i\gamma} \frac{K_-(s)\, \mathscr{S}_+(s)}{s-\xi}\, ds \tag{4.31}$$

and is analytic in the upper half-plane, $\text{Im}(\xi) > -\gamma$, and that $Q_-(\xi)$ is given by

$$Q_-(\xi) = -\frac{1}{2\pi i} \int_{-\infty + i\gamma}^{\infty + i\gamma} \frac{K_-(s)\, \mathscr{S}_+(s)}{s-\xi}\, ds \tag{4.32}$$

and is analytic in the lower half-plane, $\text{Im}(\xi) < \gamma$.

C. EVALUATION OF THE ENTIRE FUNCTION E

Our problem now is to determine the form of $E(\xi)$, given by (4.20), in the upper half-plane, $\text{Im}(\xi) > -\gamma$, and in the lower half-plane, $\text{Im}(\xi) < \gamma$. By equations (4.21), (4.28) and (4.31) we observe that $E(\xi)$ for the upper half-plane is given by

$$E(\xi) = (\xi^2 + z^2)\, \exp\left\{ \frac{1}{2\pi i} \int_{-\infty - i\gamma}^{\infty - i\gamma} \ln\left[\left(\frac{s^2+1}{s^2+z^2}\right)\left(1 - \frac{c}{s}\tan^{-1} s\right)\right] \frac{ds}{s-\xi} \right\} P(\xi) - \frac{1}{2\pi i} \int_{-\infty - i\gamma}^{\infty - i\gamma} \frac{K_-(s)\, \mathscr{S}_+(s)}{s-\xi}\, ds \,. \tag{4.33}$$

We observe that as $\xi\to\infty$ in the upper half-plane the exponential term in (4.32) goes to constant 1, and $R_+(\xi)$ goes to zero like $1/\xi$. For the function $P(\xi)$ defined by equation (4.5) we find that if $\rho(x)$ has a Taylor series expansion near zero,

$$\rho(x) = a + bx + \ldots , \tag{4.34}$$

and calculate the Fourier transform of ρ from (4.34), $P(\xi)$ will act like

$$P(\xi) = \frac{a}{\xi} + \frac{b}{\xi^2} = \frac{1}{\xi}\left(a + \frac{b}{\xi}\right) \tag{4.35}$$

as $\xi\to\infty$. Therefore, we conclude that in the upper half-plane, $\mathrm{Im}(\xi) > -\gamma$, $E(\xi)$ will act like

$$E(\xi) = a\xi + b \tag{4.36}$$

as $\xi\to\infty$.

From the conclusions of the above discussion we may conclude that the entire function $E(\xi)$ will act like

$$E(\xi) = b + a\xi + c e^{-i\xi n} P_k(\xi) \tag{4.37}$$

as $|\xi|\to\infty$, where c and n are constants, and $P_k(\xi)$ is a polynomial in ξ.

We now investigate the action of $E(\xi)$ in the lower half-plane, $\mathrm{Im}(\xi)<\gamma$, from equation (4.22). We notice that like $R_+(\xi)$ in the upper half-plane, $Q_-(\xi)$ will act like $1/\xi$ as $\xi\to\infty$ in the lower half-plane. Also, the exponential term in $K_-(\xi)$ will act like the constant 1 and $\xi\to\infty$ in the lower half-plane. Therefore, as $\xi\to\infty$ in the lower half-plane we have

$$E(\xi) = K_-(\xi)\, G_-(\xi) = (\xi^2+1)\; G_-(\xi) \; . \qquad (4.38)$$

Comparing (4.38) with (4.37) for $\mathrm{Re}\,\xi<0$ and observing that due to the monotonic properties of the E_1 kernel the term $K_-(\xi)\; G_-(\xi)$ in the lower half-plane does not act like an exponential term, we may conclude that $c\equiv 0$ in (4.37).

Therefore, we have concluded that $E(\xi)$ in the entire complex plane acts like

$$E(\xi) = b + a\xi \quad . \qquad (4.39)$$

From (4.20) and (4.39) we now obtain the equations

$$P(\xi) = \frac{b + a\xi + R_+(\xi)}{K_+(\xi)} \qquad (4.40)$$

and

$$G_-(\xi) = \frac{b + a\xi - Q_-(\xi)}{K_-(\xi)} \quad . \qquad (4.41)$$

In summary, we have taken our original equation (4.1) with the defining condition (4.2) and have reduced it to the pair of equations (4.3) and (4.4). From these equations we have applied Fourier transforms, the general Weiner-Hopf technique, and the principle of analytic continuation to obtain the entire function $E(\xi)$ in equation (4.33). Evaluating this function we have obtained explicit expressions for $P(\xi)$ and $G_-(\xi)$, given by (4.40) and (4.41). At this point we will need to know enough information to determine $P(\xi)$ and $G_-(\xi)$ at two specific values of ξ without using (4.40) and (4.41).

This will enable us to solve for the constants a and b. Having done this we can obtain expressions for our density functions $\rho(x)$ and $g(x)$ by inversion of the transforms $P(\xi)$ and $G_-(\xi)$.

This transform inversion of $P(\xi)$ and $G_-(\xi)$ often poses considerable difficulties. For example, in attempting to invert $P(\xi)$ to obtain $\rho(x)$ we find that the integrals $K_+(\xi)$ and $R_+(\xi)$ are themselves not easy to evaluate. The total density function ρ given by

$$\rho(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} P(\xi)\, d\xi \tag{4.42}$$

is itself another integral which must be evaluated. The only feasible way to proceed to evaluate (4.42) is to first evaluate $K_+(\xi)$ explicitly since it is in the denominator in equation (4.40). We then would try to interchange the order of integration in (4.42) to hopefully reduce $\rho(x)$ to a singular integral. The work of Case and Mullikin tend to imply that this can be done, but the details are at the best not obvious. A specific case where some of the integrals are evaluated is presented in Appendix D.

Since the Weiner-Hopf method was the first devised, attempts were made to actually solve problems this way [12]. It is the most elegant method and, given a certain understanding of complex function theory, is the easiest to get most of the way through -- up to the evaluation of the entire function E. The difficulties encountered from that point on make it the least popular method, now that other approaches are available.

V. CONCLUSIONS

Under the assumptions of one speed, steady state, isotropic scattering, and homogeneous media with plane symmetry, this thesis developes the complete solution of the one-dimensional neutron transport equation by three separate techniques. At first glance these methods appear to be entirely different, but are instead closely related.

The method of K. M. Case attacks the integro-differential equation directly. He considers an analog to the classical differential equation approach and uses a semi-classical eigenfunction expansion with both a continuous spectrum and a finite discrete spectrum. The former requires use of "generalized functions."

The method of T. W. Mullikin attacks the integral equation form of the neutron transport equation. He temporarily sacrifices information about the angular density, which he is able to regain after obtaining a solution for the total neutron density. In developing his solution he realizes that he cannot obtain a satisfactory solution by a straight Neumann series approach due to poor convergence properties of the series for this case. Instead, he does something which is not an obvious course of action. He converts his elementary integral equation to a singular integral equation and obtains a final result which does have good convergence properties and lends itself well to machine computation.

The Weiner-Hopf technique also attempts to solve the integral form of the transport equation. It separates the

solution into two parts, one corresponding to a solution within the medium and another corresponding to a solution outside it. Both parts of the solution are necessary to imbed the problem within a fully infinite medium in order to apply Fourier transforms. This technique depends heavily on complex variable theory from beginning to end, whereas in the others the complex variable theory enters into the solution only at the end when a Hilbert problem is solved.

The mathematics involved in obtaining these three solutions are different but interrelated. All three methods begin with entirely different approaches, but all eventually reduce to solving a Hilbert problem in the complex plane. Similar and sometimes identical functions appear within the separate developments. This is especially noticeable in the Case and Mullikin developments. The Λ and X functions are examples. At the end these two methods also have the same homogeneous and inhomogeneous Hilbert problems and essentually the same steps, though not in the same order. In the Weiner-Hopf method the similarities between it and the other two are not so noticeable. Here the Λ and X functions are concealed within other functions like $R_+(\xi)$, $Q_-(\xi)$ and $K(\xi)$.

Appendix C gives an example of how Mullikin's and Case's method can be shown to yield identical solutions. The solution due to the Weiner-Hopf technique is the same as the others, but the verification of this fact is beyond the scope of this thesis.

The principal advantage of Case's method lies in its versatility. This is due to the fact that when solving any problem this way you are essentially setting up a basis for a Hilbert Space. Therefore, any problem with a proper set of boundary conditions can be theoretically solved this way. Also, since it is a method derived from classical differential equation theory, there are many classical problems from which information may be drawn. Solutions by this method are most applicable to infinite or semi-infinite media, such as full-space or half-space problems. Solutions to finite slab problems become more difficult to obtain due to the existence of multiple sets of boundary conditions.

The primary disadvantage of this method comes from the fact that it is often very tedious and requires a proper guess of the form of the eigenfunctions in order to obtain a complete set. Once these eigenfunctions are known, however, the method is not difficult to develop. Another disadvantage to this method arises from the fact that you are automatically dealing with an infinite series, whose convergence properties are often poor from a numerical point of view. For these two reasons this method is usually considered as an academic solution to the problem and is not usually desirable for obtaining actual numerical answers.

Due to its desirable numerical convergence properties Mullikin's method is the best for obtaining numerical solutions and is especially desirable for finite slab problems. It can in fact be used for solving a variety of problems

whose solutions can be obtained in terms of solutions to slab problems. As it must, this method will generate the eigenfunctions, perhaps disguised, which have to be guessed when using Case's method (Ref. Appendix C). Thus, this method has the convenient property of providing information about the spectrum of the operator more directly than can be readily obtained by investigating Case's method.

The principal disadvantage of this method is that it is somewhat limited in scope. It is useful mainly for solving problems in finite media. Also, a frequent requirement for a solution by this method is the existence of a somewhat special type of scattering kernel. Finally, the actual algebraic manipulations about halfway through the development become somewhat involved.

The primary value of the Weiner-Hopf method lies in its historical significance, since it is a somewhat difficult method to use in practice. It was the first method developed and mathematically the most elegant. It paved the way for the development of later methods. This method convinced mathematicians and physicists of the necessity of transforming the problem into the complex plane. It also pointed out the importance of the Hilbert problem in obtaining actual solutions. This historical value of this method is being shown again as current investigations of possible methods of solving quarter-space problems are being centered first

around this technique.[1] It therefore provides valuable insight into possible approaches which might be physically more practical.

The method itself is straightforward, until it is time to evaluate some of the significant transform functions and to invert the total density transform function. At this point the method becomes very involved and hence is a difficult method from which to obtain solutions. For this reason it is unpopular as a practical tool for solving specific problems.

---

[1] Lam, S. K. and Leonard, A., Milne's Problem for Two-Dimensional Transport in a Quarter Space, Nuclear Engineering Division, Stanford University, Stanford, California, 1970.

Leonard, A., Two-Dimensional Quarter Space Problems in One-Speed Transport Theory, Nuclear Engineering Division, Stanford University, Stanford, California, 1970.

APPENDIX A: SOLUTION FOR THE INFINITE MEDIUM GREEN'S FUNCTION BY THE METHOD OF CASE

In this problem we assume we have a plane source located at x=0 and emitting in direction μ_0. We are looking for the Green's function solution to the transport equation,

$$\mu \frac{\partial}{\partial x} G(x,\mu) + G(x,\mu) = \frac{c}{2}\int_{-1}^{1} G(x,\mu')\,d\mu' + S(x,\mu), \tag{A.1}$$

where $S(x,\mu)$ is given by

$$S(x,\mu) = \frac{\delta(x)\;\delta(\mu-\mu_0)}{2\pi} . \tag{A.2}$$

This inhomogeneous equation reduces to an equivalent form consisting of the homogeneous equation,

$$\mu \frac{\partial}{\partial x} G(x,\mu) + G(x,\mu) = \frac{c}{2}\int_{-1}^{1} G(x,\mu')\,d\mu', \tag{A.3}$$

plus the boundary condition,

$$G(0^+,\mu) - G(0^-,\mu) = \frac{1}{\mu}\;\frac{\delta(\mu-\mu_0)}{2\pi} \tag{A.4}$$

Here $G(0^+,\mu)$ and $G(0^-,\mu)$ are the values of the Green's function as we approach the plane source from the right and left sides respectively.

To solve this problem we look for solutions of the form

$$\psi_\nu(x,\mu) = \phi_\nu(\mu)\, e^{-x/\nu} . \tag{A.5}$$

Therefore, applying the half-range completeness theorem, [Theorem 2.7], we wish to write the Green's function solution as

$$G(x,\mu) = a_{0+}\psi_{0+}(x,\mu) + \int_0^1 A(\nu)\psi_\nu(x,\mu)\,d\nu\,,\quad x>0, \tag{A.6}$$

and

$$G(x,\mu) = -a_{0-}\psi_{0-}(x,\mu) - \int_{-1}^0 A(\nu)\psi_\nu(x,\mu)\,d\nu\,,\quad x<0. \tag{A.7}$$

Since the right hand sides of (A.6) and (A.7) are just linear combinations of elementary solutions $\phi_\nu\, e^{-x/\nu}$, they satisfy the transport equation (A.1).

To ensure that the boundary condition is satisfied we let x approach zero in (A.6) and (A.7) and obtain

$$G(0^+,\mu) = a_{0+}\phi_{0+}(\mu) + \int_0^1 A(\nu)\,\phi_\nu(\mu)\,d\nu \tag{A.8}$$

and

$$G(0^-,\mu) = -a_{0-}\phi_{0-}(\mu) - \int_{-1}^0 A(\nu)\,\phi_\nu(\mu)\,d\nu\,. \tag{A.9}$$

We now combine (A.4), (A.8), and (A.9) to get

$$G(0^+,\mu) - G(0^-,\mu) = \frac{\delta(\mu-\mu_0)}{2\pi\mu} = a_{0+}\phi_{0+}(\mu) + a_{0-}\phi_{0-}(\mu) + \int_{-1}^1 A(\nu)\phi_\nu(\mu)\,d\nu. \tag{A.10}$$

If we now solve (A.10) for $a_{0\pm}$, $\phi_{0\pm}(\mu)$, $A(\nu)$ and $\phi_\nu(\mu)$ by the formulas obtained in II we have as our infinite medium Green's function solution

$$G(x,\mu) = \pm\frac{1}{2\pi}\left[\frac{\phi_{0\pm}(\mu_0)\,\phi_{0\pm}(\mu_0)\,e^{-|x-x_0|/\nu_0}}{N_{0\pm}} + \int_{0\pm}^1 \frac{\phi_{\pm\nu}(\mu)\,\phi_{\pm\nu}(\mu_0)\,e^{-|x-x_0|/\nu}}{N(\pm\nu)}\,d\nu\right] \tag{A.11}$$

where the principle of translational invariance allows us to shift the source to x_0, and the upper signs apply for $x>x_0$, the lower for $x<x_0$.

APPENDIX B: SOLUTION OF THE ALBEDO PROBLEM BY CASE'S METHOD

The physical problem usually referred to as the "albedo problem" is that of obtaining the angular neutron density everywhere in an internally source-free half-space, $0 \leq x \leq \infty$, if a parallel beam of neutrons is incident on the surface at x=0.

Let $\psi_a(x,\mu)$ be the solution for the albedo problem. The mathematical interpretation of a parallel beam of neutrons incident on the surface at x=0 takes the form of the boundary condition

$$\psi_a(0,\mu) = \delta(\mu-\mu_0) , \quad \mu_0,\mu > 0 . \tag{B.1}$$

Another boundary condition common to almost all infinite half-space problems comes from the fact that at $x=\infty$ it is desired that the neutron angular density be negligible, or in other words

$$\lim_{x\to\infty} \psi_a(x,\mu) = 0 . \tag{B.2}$$

As before we look for solutions of the form

$$\psi_\nu(x,\mu) = \phi_\nu(\mu)\, e^{-x/\nu} , \tag{B.3}$$

and hence applying half-range completeness the most general solution to the homogeneous transport equation becomes

$$\psi_a(x,\mu) = a_{0+}\, \psi_{0+}(x,\mu) + \int_0^1 A(\nu)\, \psi_\nu(x,\mu)\, d\nu . \tag{B.4}$$

Setting x equal to zero and applying (B.1) and (B.3) yields

$$\delta(\mu-\mu_0) = a_{0+}\phi_{0+}(\mu) + \int_0^1 A(\nu)\,\phi_\nu(\mu)\,d\nu\,, \quad \mu \geq 0. \tag{B.5}$$

Multiplying (B.5) by $W(\mu)\ \phi_\nu(\mu)$ and integrating over $\mu \geq 0$, we obtain the equation

$$\begin{aligned}\int_0^1 W(\mu)\,\phi_\nu(\mu)\,\delta(\mu-\mu_0)\,d\mu &= a_{0+}\int_0^1 \phi_{0+}(\mu)\,\phi_\nu(\mu)\,W(\mu)\,d\mu \\ &+ \int_0^1\!\!\int_0^1 W(\mu)\,\phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,A(\nu')\,d\nu'\,d\mu\,, \quad 0 \leq \nu,\nu' \leq 1.\end{aligned} \tag{B.6}$$

By the principle of half-range orthogonality [equation (2.87)], the first integral on the right hand side of (B.6) vanishes. Evaluation of the normalization integral within the second right hand term gives

$$\int_0^1 W(\mu)\,\phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,d\mu = W(\nu)\,\frac{N(\nu)}{\nu}\,\delta(\nu-\nu')\,. \tag{B.7}$$

Therefore, equation (B.6) can be reduced to

$$W(\mu_0)\ \phi_\nu(\mu_0) = W(\nu)\ \frac{N(\nu)}{\nu}\,A(\nu)\,, \tag{B.8}$$

from which we obtain an expression for $A(\nu)$

$$A(\nu) = \frac{\nu\,W(\mu_0)\ \phi_\nu(\mu_0)}{N(\nu)\ W(\nu)}\,. \tag{B.9}$$

Multiplying (B.5) by $W(\mu)\cdot\phi_{0+}(\mu)$ and integrating over $\mu \geq 0$, we get

$$\begin{aligned}\int_0^1 W(\mu)\,\phi_{0+}(\mu)\,\delta(\mu-\mu_0)\,d\mu &= \int_0^1 a_{0+}\,\phi_{0+}(\mu)\,W(\mu)\,\phi_{0+}(\mu)\,d\mu \\ &+ \int_0^1\!\!\int_0^1 A(\nu')\,\phi_{\nu'}(\mu)\,W(\mu)\,\phi_{0+}(\mu)\,d\nu'\,d\mu\,, \quad 0 \leq \nu,\nu' \leq 1\end{aligned} \tag{B.10}$$

Due to half-range orthogonality the second term on the right hand side of (B.10) vanishes. The first term on the right hand side is the normalization integral

$$\int_0^1 a_{0+} W(\mu)\, \phi_{0+}(\mu)\, d\mu = -\left(\frac{c\nu_0}{2}\right)^2 X(\nu_0)\, a_{0+} \tag{B.11}$$

Therefore, equation (B.10) reduces to

$$W(\mu_0)\, \phi_{0+}(\mu_0) = -\left(\frac{c\nu_0}{2}\right)^2 X(\nu_0)\, a_{0+}\,, \tag{B.12}$$

from which we obtain an expression for a_{0+}

$$a_{0+} = -\frac{4}{(c\nu_0)^2}\, \frac{W(\mu_0)\, \phi_{0+}(\mu_0)}{X(\nu_0)}\,. \tag{B.13}$$

Our complete solution to the albedo problem now becomes

$$\begin{aligned} \Psi_d(x,\mu) = & -\frac{4}{(c\nu_0)^2}\, \frac{W(\mu_0)}{X(\nu_0)}\, \phi_{0+}(\mu_0)\, \phi_{0+}(\mu)\, e^{-x/\nu_0} \\ & + W(\mu_0) \int_0^1 \frac{\nu\, \phi_\nu(\mu_0)\, \phi_\nu(\mu)}{N(\nu)\, W(\nu)}\, e^{-x/\nu}\, d\nu\,. \end{aligned} \tag{B.14}$$

APPENDIX C: SOLUTION OF THE ALBEDO PROBLEM BY MULLIKIN'S METHOD

For the albedo problem, as described in Appendix B, the only source of neutrons is due to a unidirectional beam of neutrons incident at the x=0 surface of a half-space. Since this description of the source of neutrons corresponds exactly with that used in III to develop Mullikin's method, we can take as our uncollided source function $\overline{S}(x)$ for the albedo problem the same as that used in III, namely

$$\overline{S}(x) = e^{-x/\mu_0} . \tag{C.1}$$

We conclude that the solution for the albedo problem will be given by the results of III if we let the slab thickness τ become infinitely large. Therefore, if we truncate the Neumann series solutions for the functions $h_{1(\pm)}(x,z)$ and $h_{2(\pm)}(z)$ in equations (3.51) and (3.52), we can apply these results to determine an approximate solution for $J(x,z)$.

The Neumann series solutions for $h_{1(\pm)}(x,z)$ and $h_{2(\pm)}(z)$ are given by the operator equations

$$\begin{aligned} h_{1(\pm)}(x,z) &= (I + (\pm)\mathcal{L})^{-1}\$_{(\pm)}(x,z) \\ &= (I - (\pm)\mathcal{L} + (\pm)\mathcal{L}^2 - (\pm)\mathcal{L}^3 + \ldots)\,\$_{(\pm)}(x,z) \\ &= \$_{(\pm)}(x,z) - (\pm)\mathcal{L}(\$_{(\pm)}(x,z)) + (\pm)\mathcal{L}^2(\$_{(\pm)}(x,z)) - \ldots \end{aligned} \tag{C.2}$$

$$\begin{aligned} h_{2(\pm)}(z) &= (I + (\pm)\mathcal{L})^{-1} e^{-\tau/z} z X(-z) \\ &= e^{-\tau/z} z X(-z) - (\pm)\mathcal{L}(e^{-\tau/z} z X(-z)) + \ldots \end{aligned} \tag{C.3}$$

To simplify calculations let us take the following approximations for $h_{1(\pm)}(x,z)$ and $h_{2(\pm)}(z)$ by truncating the Neumann series expressions (C.2) and (C.3):

$$h_{1(\pm)}(x,z) = \$_{(\pm)}(x,z) - (\pm)\int_0^1 H(z,t)\, \frac{\$_{(\pm)}(x,t)}{t+z}\, dt \qquad (C.4)$$

and

$$h_{2(\pm)}(z) = e^{-\tau/z}\, z\, X(-z), \qquad (C.5)$$

where H(z,t) is the kernel of the integral operator L,

$$H(z,t) = \frac{c}{2}\, \frac{z\, X(-z)}{X^{+}(t)\, \Lambda^{-}(t)}\, e^{-\tau/z} \quad . \qquad (C.6)$$

We now ask the question, for what values of c will we have subcritical results. Referring to equation (3.53) we see that as τ becomes infinitely large the criticality condition for the half-space will be that the denominator of $C_{(\pm)}(x)$ approach zero. If we use the approximation of $h_{2(\pm)}(z_0)$ given by (C.5), we see that the denominator of $C_{(\pm)}(x)$ will vanish when the following equation is satisfied,

$$(\pm)e^{-\tau/z_0} = \frac{X(z_0)}{X(-z_0)} = \left(\frac{1+z_0}{1-z_0}\right) \exp\left\{\frac{1}{2\pi}\int_c^1 \theta(t)\left[\frac{1}{t-z_0} - \frac{1}{t+z_0}\right]dt\right\},\ \tau\to\infty. \qquad (C.7)$$

However, this will in fact happen only when $z_0\to\infty$, where z_0 is real, or in other words when $c\to 1$, if c is less than 1 (ref. Chapter II).

When c is taken as greater than one, we know from Chapter II that z_0 will have the form $z_0 = i\omega_0$. If we apply the defining relation

$$\frac{X(-i\omega_0)}{X(i\omega_0)} \equiv -\, e^{-2\epsilon/i\omega_0}, \qquad (C.8)$$

the criticality condition then takes the form

$$-e^{2\epsilon/i\omega_0} = (\pm)\, e^{-\tau/i\omega_0} \quad , \tau\to\infty , \qquad (C.9)$$

or

$$2\epsilon + \tau = \pi\omega_0 \quad , \quad \tau\to\infty . \qquad (C.10)$$

Considering the last term in (C.7) and the above equation (C.9), we obtain the equation

$$e^{-i\pi + i\,2\epsilon/\omega_0} = \left(\frac{1-i\omega_c}{1+i\omega_c}\right) \exp\left\{\frac{1}{2\pi}\int_0^1 \theta(t)\left[\frac{1}{1+i\omega_c} - \frac{1}{1-i\omega_c}\right] dt\right\}, \qquad (C.11)$$

which we can solve for ε to determine the following result,

$$\lim_{c\to 1^+} \epsilon < \infty \, . \qquad (C.12)$$

Therefore, the condition for criticality, (C.10), when $c>1$ is that $\omega_0\to\infty$ as $\tau\to\infty$, which implies that $c\to 1^+$. Therefore, we conclude that, unlike the finite slab case when $c\geq 1$, it is impossible to have a subcritical result in the case of the half-space when $c\geq 1$. For this reason we will proceed with our solution to the albedo problem considering $c<1$ and hence z_0 a real number (ref. Chapter II).

From equation (3.53) we see that $f_{(\pm)}(x,z)$ takes the form

$$f_{(\pm)}(x,z) = \frac{1}{\Lambda(z)}\Big\{-(\pm)\int_0^1 H(z,t)\,\frac{\phi_{(\pm)}(x,t)}{t+z}\,dt - e^{-\tau/z}\int_0^1 H(-z,t)\,\frac{\phi_{(\mp)}(x,t)}{t-z}\,dt$$
$$+ \phi_{(\pm)}(x,z) + C_{(\pm)}(x)\left[e^{-\tau/z}\, z\, X(-z) - (\pm)\, z\, X(z)\right]\Big\}, \qquad (C.13)$$

and from (3.54) we determine $C_{(\pm)}(x)$ as

$$C_{(\pm)}(x) = -\frac{1}{[e^{-\tau/z_0} z_0 X(-z_0) - (\pm) z_0 X(z_0)]} \Big\{ -(\pm) \int_0^1 H(z_0,t) \frac{\$_{(\pm)}(x,t)}{t+z_0} dt \tag{C.14}$$
$$- e^{-\tau/z_0} \int_0^1 H(-z_0,t) \frac{\$_{(\pm)}(x,t)}{t-z_0} dt + \$_{(\pm)}(x,z_0) \Big\}$$

where z_0 is a real zero of $\Lambda(z)$ and $c<1$. If we now introduce the explicit form of $H(z,t)$ given by (C.6) into (C.7) and (C.8) and take the limit of $f_{(\pm)}(x,z)$ as τ approaches positive infinity, we obtain

$$\bar{f}_{(\pm)}(x,z) = \lim_{\tau\to\infty} f_{(\pm)}(x,z) = \frac{1}{\Lambda(z)} \Big\{ \frac{c}{2} z X(z) \int_c^1 \frac{e^{-x/t}}{X^+(t)\Lambda^-(t)} \frac{dt}{t-z} \tag{C.15}$$
$$+ e^{-x/z} - \frac{z X(z)}{z_0 X(z_0)} \Big[\frac{c}{2} z_0 X(z_0) \int_c^1 \frac{e^{-x/t}}{X^+(t)\Lambda^-(t)} \frac{dt}{t-z_0} + e^{-x/z_0} \Big] \Big\}.$$

To determine our solution for $J(x,\mu_0)=\rho(x)$ from (C.15) we must let z approach μ_0 on the cut from below, yielding $\bar{f}_-$, and from above, yielding $\bar{f}_+$, and apply the relation

$$\rho(x) = J(x,\mu_0) = \frac{1}{2}\left[\bar{f}_+(x,\mu_0) + \bar{f}_-(x,\mu_0)\right]. \tag{C.16}$$

Our solution then takes the form

$$\rho(x) = \frac{1}{2} e^{-x/\mu_0} \left[\frac{1}{\Lambda^+(\mu_0)} + \frac{1}{\Lambda^-(\mu_0)}\right] + \frac{\mu_0 X^+(\mu_0)}{\Lambda^+(\mu_0)} \frac{c}{2} \int_c^1 \frac{e^{-x/\nu}}{X^+(\nu)\Lambda^-(\nu)} \frac{d\nu}{\nu-\nu_0} \tag{C.17}$$
$$- \frac{\mu_0}{\nu_0} \frac{X^+(\mu_0)}{X(\nu_0)\Lambda^+(\mu_0)} e^{-x/\nu_0} - \frac{\mu_0 X^+(\mu_0)}{\Lambda^+(\mu_0)} \frac{c}{2} \int_c^1 \frac{e^{-x/\nu}}{X^+(\nu)\Lambda^-(\nu)} \frac{d\nu}{\nu-\nu_0},$$

where we have set $z_0=\nu_0$ to conform to the notation used in Chapter II.

Let us now compare this solution (C.17) with that obtained by Case's method, (B.14). If we substitute the

appropriate definitions for the functions W, N, and ϕ_{0+} into (B.14), integrate over μ, and apply our normalization (2.5), we obtain the following result,

$$\rho(x) = -\frac{\mu_0 X^+(\mu_0)}{\nu_0 X(\nu_0)\Lambda^+(\mu_0)} e^{-x/\nu_0} \tag{C.18}$$
$$+ \int_0^1 \frac{\mu_0 X^+(\mu_0)\Lambda^+(\nu)(\nu_0-\mu_0)\phi_\nu(\mu_0)}{\nu\, \Lambda^+(\mu_0)\Lambda^-(\nu) X^+(\nu)(\nu_0-\nu)} e^{-x/\nu} d\nu,$$

We observe that the third term in Mullikin's solution (C.17) corresponds exactly with the first term in Case's solution (C.18). We recall that $\phi_\nu(\mu_0)$ in (C.18) has a principal value term and a delta function term. If we combine the second and fourth term of (C.17) by applying the identity

$$\frac{1}{\nu-\mu_0} - \frac{1}{\nu-\nu_0} = \frac{\mu_0-\nu_0}{(\nu-\mu_0)(\nu-\nu_0)} \tag{C.19}$$

and interpreting the integration as principal value integration, we obtain the principal value integral part of the second term in (C.18). We can integrate the delta function part directly to obtain the first term of (C.17). We conclude that the two solutions for the albedo problem are in fact different expressions for the same functions.

APPENDIX D: SOLUTION OF THE ALBEDO PROBLEM BY THE WEINER-HOPF TECHNIQUE

As shown in Appendices B and C, for the albedo problem the uncollided source function S(x) in equation (4.3) takes the form

$$S(x) = e^{-x/\mu_0}, \tag{D.1}$$

where $\mu_0 = \cos\theta_0$ and θ_0 is the angle that the beam makes with the surface normal at x=0. Using (D.1), the Fourier transform for the uncollided source exists and takes the form

$$\mathcal{S}_+(\xi) = \frac{\mu_0}{1 - i\xi\mu_0}, \tag{D.2}$$

using residue theory to compute the integral. We notice from (D.2) that the singularity of $\mathcal{S}_+(\xi)$ is at $\xi = -i/\mu_0$.

Our problem now is to evaluate $K_+(\xi)$, $K_-(\xi)$ and $R_+(\xi)$, given by (4.28), (4.29) and (4.31), respectively. Having done this and having previously determined values for the constants a and b, we can find an exact expression for $P(\xi)$ by (4.40). We then can determine $\rho(x)$ using (4.42), the standard inverse integral.

To evaluate $K_+(\xi)$ let us first investigate the function $\ln K_+^*(\xi)$ given by (4.27),

$$\ln K_+^*(\xi) = \frac{1}{2\pi i}\int_{-\infty - i\gamma}^{\infty - i\gamma} \ln\left[\left(\frac{s^2+1}{s^2+z^2}\right)\left(1 - K(s)\right)\right]\frac{ds}{s-\xi}\ . \tag{D.3}$$

From the form of the function (1-K(s)) given by (4.16) in terms of logarithms, we observe that the integrand of (D.3) has branch lines $[i, i\infty)$ and $[-i, -i\infty)$, due to the branches

of K(s). The integrand being itself a logarithm has branches from its zeros, which are again $[i,i\infty)$ and $[-i,-i\infty)$, since its only zeros are at $\pm i$. Let us now apply Cauchy's integral formula to (D.3), where we have closed up the integral of (D.3) for the upper half-plane in the usual manner. Hence our integration contour is composed of the original line $(-\infty-i\gamma,\ \infty-i\gamma)$, two half-arcs at ∞, an integral down the positive side of the upper branch line, and an integral up the negative side of the upper branch line. Applying the Cauchy integral formula we obtain the result

$$\ln K_+^*(\xi) = \ln\left[\left(\frac{\xi^2+1}{\xi^2+Z^2}\right)(1-K(\xi))\right] + \frac{1}{2\pi i} I_P - \frac{1}{2\pi i} I_N \qquad \text{(D.4)}$$

where I_P and I_N are the integrals up the positive and negative sides of the branch line, respectively, given by

$$I = \int_i^{i\infty} \ln\left[\left(\frac{s^2+1}{s^2+Z^2}\right)\left(1-\frac{c}{s}\tan^{-1}s\right)\right] \frac{ds}{s-\xi} \quad . \qquad \text{(D.5)}$$

Let us now attempt to evaluate I_P and I_N. Into I_P let us substitute the identity

$$1-\frac{c}{s}\tan^{-1}s \equiv 1-\frac{ic}{2s}\ln\left(\frac{i+s}{i-s}\right) \qquad \text{(D.6)}$$

and the change of variable

$$s \equiv i + i\eta \quad . \qquad \text{(D.7)}$$

Our resulting equation takes the form

$$I_P = \int_0^{\infty} \ln\left[\left(\frac{\eta^2+2\eta}{\eta^2+2\eta+1-Z^2}\right)\left(1-\frac{c}{2(\eta+1)}\ln\left(\frac{\eta+2}{\eta}\right)+\frac{c}{2(\eta+1)}\pi i\right]\frac{d\eta}{1+\eta+i\xi}, \qquad \text{(D.8)}$$

where we have used

$$\ln\left(-\frac{\eta+2}{\eta}\right) \equiv \ln\left(\frac{\eta+2}{\eta}\right) - \pi i \tag{D.9}$$

To determine an expression for I_N we need only recall the fact that the value of the logarithm on the negative side of the branch line equals the value on the positive side plus $2\pi i$ and use (D.8),

$$I_N = \int_0^\infty \ln\left[\left(\frac{\eta^2+2\eta}{\eta^2+2\eta+1-Z^2}\right)\left(1-\frac{c}{2(\eta+1)}\ln\left(\frac{\eta+2}{\eta}\right)-\frac{c}{2(\eta+1)}\pi i\right)\right]\frac{d\eta}{1+\eta+i\xi} \tag{D.10}$$

If we now perform the subtraction indicated by (D.4) we obtain

$$I_P - I_N = \int_0^\infty \ln\left\{\frac{1-\frac{c}{2(\eta+1)}\ln\left(\frac{\eta+2}{\eta}\right)+\frac{c}{2(\eta+1)}\pi i}{1-\frac{c}{2(\eta+1)}\ln\left(\frac{\eta+2}{\eta}\right)-\frac{c}{2(\eta+1)}\pi i}\right\}\frac{d\eta}{1+\eta+i\xi} \tag{D.11}$$

or

$$I_P - I_N = \int_0^\infty i\left(\phi_1(\eta)-\phi_2(\eta)\right)\frac{d\eta}{1+\eta+i\xi} \quad , \tag{D.12}$$

where $\phi_1(\eta)$ and $\phi_2(\eta)$ are the arguments of the numerator and denominator, respectively, in the logarithm term of the integrand. Taking the exponential of both sides of (D.4) we now have a simplified expression for $K_+^*(\xi)$,

$$K_+^*(\xi) = \left(\frac{\xi^2+1}{\xi^2+Z^2}\right)\left(1-K(\xi)\right)\exp\left\{\frac{1}{2\pi}\int_0^\infty(\phi_1(\eta)-\phi_2(\eta))\frac{d\eta}{1+\eta+i\xi}\right\}. \tag{D.13}$$

From (4.28) and (4.17) we can obtain simplified expressions for $K_+(\xi)$ and $K_-(\xi)$,

$$K_+(\xi) = (\xi^2+1)\left(1-\frac{c}{\xi}\tan^{-1}\xi\right)\exp\left\{\frac{1}{2\pi}\int_0^{\infty}(\phi_1(\eta)-\phi_2(\eta))\frac{d\eta}{1+\eta+i\xi}\right\} \tag{D.14}$$

$$K_-(\xi) = (\xi^2+1)\exp\left\{\frac{1}{2\pi}\int_0^{\infty}(\phi_1(\eta)-\phi_2(\eta))\frac{d\eta}{1+\eta+i\xi}\right\} . \tag{D.15}$$

To evaluate $R_+(\xi)$ we refer to its defining equation (4.31). We recall that the singularity of $\emptyset_+(s)$ is at $s=-i/\mu_0$ and that $K_-(s)$ is analytic in the lower half-plane. Therefore, if we close this integral in the lower half-plane and evaluate the residue at the pole $-i/\mu_0$, we obtain a simplified form for R_+,

$$R_+(\xi) = \frac{K_-(-i/\mu_0)}{\xi + i/\mu_0} \tag{D.16}$$

The function $P(\xi)$ now becomes

$$P(\xi) = \frac{b + a\xi + \left[K_-(-\frac{i}{\mu_0})/(\xi + i/\mu_0)\right]}{K_+(\xi)} \tag{D.17}$$

Let us now evaluate $\rho(x)$ from (4.42). If we take x negative and close the contour for (4.42) in the upper half-plane, we see that the integrand is analytic in this half-plane and hence $\rho(x)$ will equal zero, the result we expect. If x is positive we close the integral in the lower half-plane where $R_+(\xi)$ has a simple pole at $\xi=-i/\mu_0$, $K_+(\xi)$ has a zero at $\xi=-iz$, and $K_+(\xi)$ has the branch line $[-i,-i\infty)$. If we investigate the similarities among the zeros of the functions $(1-K(\xi))$ and $\Lambda(\nu)$ from Chapter II, we find that in the notation of Case

$$z = -i/\nu_0 \tag{D.18}$$

Applying the Cauchy integral formula again we see that the contribution to ρ at the pole $\xi=-i/\mu_0$ is

$$\frac{K_-(-i/\mu_0)}{K_+(-i/\mu_0)}\, e^{-ix(-i/\mu_0)} = \frac{K_-(-i/\mu_0)}{K_+(-i/\mu_0)}\, e^{-x/\mu_0} . \tag{D.19}$$

The contribution to ρ at the zero, $-iz$, of K_+ is

$$\frac{b - a\, i/\nu_0 + \left[K_-(-i/\mu_0) \big/ i\left(\frac{1}{\mu_0} - \frac{1}{\nu_0}\right)\right]}{\left[1 - \left(\frac{1}{\nu_0}\right)^2\right] \exp\left\{\frac{1}{2\pi}\int_0^\infty (\phi_1(\eta) - \phi_2(\eta))\frac{d\eta}{1+\eta+1/\nu_0}\right\}}\, e^{-x/\nu_0} .$$

In a similar manner to (D.10) above we can reduce the two branch integrals to integrals from 1 to ∞ with respect to ε if we set $\xi=-i\varepsilon$. Denoting these integrals on the positive and negative sides of the branch line by I_1^* and I_2^*, respectively, we obtain the resulting expression for ρ,

$$\rho(x) = \frac{K_-(-i/\mu_0)}{K_+(-i/\mu_0)}\, e^{-x/\mu_0} + \frac{\left[b - a\, i/\nu_0 - \dfrac{i\, K_-(-i/\mu_0)}{(1/\mu_0 - 1/\nu_0)}\right]}{\left[1 - 1/\nu_0^2\right] \exp\left\{\frac{1}{2\pi}\int_0^\infty (\phi_1(\eta) - \phi_2(\eta))\frac{d\eta}{1+\eta+1/\nu_0}\right\}}\, e^{-x/\nu_0} + I_1^* - I_2^* . \tag{D.20}$$

Referring to the results of Appendices B and C we notice that the format of (D.20) is quite similar to the results from the other two methods since (D.20) contains an exponential term in e^{-x/ν_0}, an exponential term in e^{-x/μ_0}, and a pair of principal value integrals from 0 to 1 (remembering that $0<\nu<1$ corresponds to $1<\varepsilon<\infty$). We expect that with further

manipulation the form (D.20) can be shown to correspond to the previous forms. To do this, however, involves techniques beyond the scope of this paper and the talents of the author.

BIBLIOGRAPHY

1. Case, K. M., "On the Boundary Value Problems of Linear Transport Theory," Transport Theory, v. 1, p. 17-36, American Mathematical Society, 1969.

2. Case, K. M. and Zweifel, P. F., Linear Transport Theory, Addison-Wesley, 1967.

3. Churchill, R. V., Complex Variables and Applications, 2d ed., McGraw-Hill, 1960.

4. Dettman, J. W., Mathematical Methods in Physics and Engineering, 2d ed., p. 393-399, McGraw-Hill, 1969.

5. Garrettson, G. A., Green's Functions for Multidimensional Neutron Transport in a Slab, Ph.D. Thesis, Stanford University, Stanford, California, 1969.

6. Leonard, A. and Mullikin, T. W., "Spectral Analysis of the Anisotropic Neutron Transport Kernel in Slab Geometry with Applications," Journal of Mathematics and Physics, v. 5, p. 399-411, March 1964.

7. Leonard, A. and Mullikin, T. W., "Integral Equations with Difference Kernels on Finite Intervals," Transactions of the American Mathematical Society, v. 116, p. 465-472, April 1965.

8. Meghreblian, R. V. and Holmes, D. K., Reactor Analysis, p. 330-417, McGraw-Hill, 1960.

9. Mikhlin, S. G., Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology, 2d ed., p. 126-131, Macmillan, 1964.

10. Morse, P. M. and Feshbach, H., Methods of Theoretical Physics, v. 2, p. 1625-1627, McGraw-Hill, 1953.

11. Mullikin, T. W., "Nonlinear Integral Equations of Radiative Transfer," Nonlinear Integral Equations, v. 11, p. 345-374, University of Wisconsin Press, 1964.

12. Noble, B., Methods Based on the Weiner-Hopf Technique for the Solution of Partial Differential Equations, Pergamon Press, 1958.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Assoc Professor C. Comstock, Code 53Zk Department of Mathematics Naval Postgraduate School Monterey, California 93940	1
4.	Asst Professor G. A. Garrettson, Code 61Gr Department of Physics Naval Postgraduate School Monterey, California 93940	1
5.	Doctor R. W. Preisendorfer 2525 Correa Road Honolulu, Hawaii, 96822	1
6.	ENSIGN Larry Thomas Davis, USN 1516 East Geer Street Durham, North Carolina 27704	1

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

MATHEMATICAL SOLUTIONS OF THE ONE-DIMENSIONAL NEUTRON TRANSPORT EQUATION

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Master's Thesis, June 1971

5. AUTHOR(S) (First name, middle initial, last name)

Larry T. Davis

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1971	93	12

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

Considering the case of one speed, steady state, isotropic scattering in homogeneous media with plane symmetry, this thesis developes the solution of the one-dimensional neutron transport equation by three separate techniques. The method of K. M. Case which makes use of the theory of generalized functions in forming a semi-classical eigenfunction expansion with both a continuous spectrum and a finite discrete spectrum is developed. Converting the neutron transport equation to an integral equation and then to a singular integral equation, a solution is found in a method due to T. W. Mullikin which has very useful convergence properties. Applying the method due to N. Weiner and E. Hopf to the integral equation form of the neutron transport equation, a solution is developed which depends heavily on complex variable theory. The similarities, differences, advantages and disadvantages in the three methods are pointed out, and specific example solutions are presented.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

UNCLASSIFIED
Security Classification

A-31408

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Neutron Transport						
Weiner-Hopf						
Singular Integral Equation						
Eigenfunctions						